FOURIER ANALYSIS & METHODS

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Abstract. Caveat Emptor! These are just informal lecture notes. Errors are inevitable! Read at your own risk! Also, this is by no means a substitute for the textbook, which is warmly recommended: *Fourier Analysis and Its Applications,* by Gerald B. Folland. He was the first math teacher I had at university, and he is awesome. A brilliant writer. So, why am I even doing this? Good question...

1. 2019.03.13

When to use what? It's time for a sound check. First, speaking of sound, let us investigate the solution we found last time for the initial value problem for the wave equation on $\mathbb{R}$.

$$\begin{cases} u(x,0) = f(x) & x \in \mathbb{R} \\ u_t(x,0) = g(x) & x \in \mathbb{R} \\ \Box u = 0 & x \in \mathbb{R},\, t > 0 \end{cases}.$$

We used the Fourier transform and its properties to obtain the solution:

$$u(x,t) = \frac{1}{2}\left(f(x+t) + f(x-t)\right) + \frac{1}{2}\int_{\mathbb{R}} g(y)\chi_t(x-y)dy.$$

We can simplify the second term a bit,

$$\frac{1}{2}\int_{\mathbb{R}} g(y)\chi_t(x-y)dy = \frac{1}{2}\int_{t-x}^{t+x} g(y)dy,$$

so that

$$u(x,t) = \frac{1}{2}\left(f(x+t) + f(x-t) + \int_{x-t}^{x+t} g(y)dy\right).$$

Now let's see what happens if we make a few assumptions about our initial data. Let's assume that f and g are both non-negative and have compact support. For example, assume that g and f are both zero outside the interval $[0, 100]$. This means that our infinite string, the real line, has got some displacement and movement contained inside the interval $[0, 100]$. Consequently, if

$$x + t < 0 \implies x - t < 0 \text{ also and so } u(x,t) = 0.$$

So in particular if

$$x < -t \implies u(x,t) = 0.$$

This means that the wave only reaches the negative real axis (like a point $x = -t$) after t units of time. Hence, the wave propagates with finite speed.

We also have that if

$$100 < x - t \implies 100 < x + t \text{ also and so } u(x,t) = 0.$$

So in particular if

$$100 + t < x \implies u(x,t) = 0.$$

This means that the wave only reaches the points past 100, like $x = 100 + t$ after t units of time. So basically the wave is traveling with unit speed.

1.1. **PDEs on a half-line.** Assume that the problem is for $x \in [0, \infty)$. To determine what method to use, look at what happens at $x = 0$.

(1) If the condition is $u(0,t) = 0$, the following method is reasonable to try: extend the initial data $f(x)$ defined on $[0, \infty)$ oddly and use the Fourier transform in the x variable. OBS! Only do this if the initial data is in $\mathcal{L}^2$.
(2) If the condition is $u_x(0,t) = 0$, the following method is reasonable to try: extend the initial data $f(x)$ defined on $[0, \infty)$ evenly and use the Fourier transform in the x variable. OBS! Only do this if the initial data is in $\mathcal{L}^2$.
(3) If the condition is $u(0,t) = f(t)$ the following method is reasonable to try: hit the PDE with the Laplace transform in the t variable. For this method you do not need $f(t)$ to be in $\mathcal{L}^2$. You do need to extend f to be zero for t negative, and you need f not to grow faster than e^{at} as $t \to \infty$ for some $a > 0$.

Let's see some examples of this. First, consider the problem

$$\begin{cases} u(0,t) = 0 & t > 0 \\ u(x,0) = f(x) & x \in (0,\infty) \\ u_t - u_{xx} = 0 & t, x > 0 \end{cases}.$$

According to my suggestions above, a reasonable method to try is to extend the initial data f oddly and use the Fourier transform. So, we do this. We define

$$F(x) = \begin{cases} f(x) & x > 0 \\ -f(-x) & x < 0 \end{cases}.$$

This function is odd, and it is equal to f on the positive real axis. That is what it means to be an odd extension.

Let us hit the PDE with the Fourier transform in the x variable:

$$\hat{u}_t(\xi, t) - \widehat{u_{xx}}(\xi, t) = 0.$$

The Fourier transform takes those x derivatives and turns them into multiplication by $i\xi$, so we have

$$\hat{u}_t(\xi, t) - (i\xi)^2 \hat{u}(\xi, t) = 0.$$

Re-arranging

$$\hat{u}_t(\xi, t) = -\xi^2 \hat{u}(\xi, t).$$

This is an ODE which we can solve:

$$\hat{u}(\xi, t) = a(\xi) e^{-\xi^2 t}.$$

To determine the coefficient function we use the *extended* initial data:

$$\hat{u}(\xi, 0) = \hat{F}(\xi).$$

Consequently:

$$a(\xi) = \hat{F}(\xi).$$

OBS! We extended f *oddly.*

Exercise 1. *Show that when you extend a function defined on* $(0, \infty)$ *oddly, and then take its Fourier transform, the Fourier transform is also odd.*

So, we have found

$$\hat{u}(\xi, t) = \hat{F}(\xi) e^{-\xi^2 t}.$$

We have found a function whose Fourier transform is $e^{-\xi^2 t}$, and this is also basically item 9 from Folland's table 2. A function whose Fourier transform is $\hat{F}(\xi)$ is our oddly extended function F. Thus, the solution is

$$\int_{-\infty}^{\infty} F(y) e^{-(x-y)^2/(4t)} (4\pi t)^{-1/2} dy.$$

We need to put this back in terms of our initial data f. To do that we split the integral into the positive real axis and the negative real axis:

$$\int_{-\infty}^{0} F(y) e^{-(x-y)^2/(4t)} (4\pi t)^{-1/2} dy + \int_{0}^{\infty} f(y) e^{-(x-y)^2/(4t)} (4\pi t)^{-1/2} dy.$$

To get the integral on the negative real line in terms of our function f we use that

$$y < 0 \implies F(y) = -f(-y).$$

So

$$\int_{-\infty}^{0} F(y) e^{-(x-y)^2/(4t)} (4\pi t)^{-1/2} dy = \int_{-\infty}^{0} -f(-y) e^{-(x-y)^2/(4t)} (4\pi t)^{-1/2} dy.$$

Next we make a substitution: $z = -y$ so $dz = -dy$ and the integral goes from ∞ to 0 in the z variable:

$$= \int_{\infty}^{0} f(z) e^{-(x+z)^2/(4t)} (4\pi t)^{-1/2} dz = -\int_{0}^{\infty} f(z) e^{-(x+z)^2/(4t)} (4\pi t)^{-1/2} dz.$$

Putting this together with the other term our solution is:

$$u(x,t) = \int_{0}^{\infty} f(y) e^{-(x-y)^2/(4t)} (4\pi t)^{-1/2} dy - \int_{0}^{\infty} f(z) e^{-(x+z)^2/(4t)} (4\pi t)^{-1/2} dz$$
$$= \int_{0}^{\infty} f(y) \left[e^{-(x-y)^2/(4t)} - e^{-(x+y)^2/(4t)} \right] (4\pi t)^{-1/2} dy.$$

Indeed, if $x = 0$ our solution vanishes, so the boundary condition is satisfied, the PDE is satisfied, and the initial condition is satisfied.

1.1.1. *DO NOT DO THIS.* What would happen if we instead tried to extend the function f to $\mathbb{R}$, defining

$$f_{bad}(x) = \begin{cases} f(x) & x > 0 \\ 0 & x < 0 \end{cases}?$$

Well, we repeat the same procedure, applying the Fourier transform in the x variable. Everything works in the same way, up to the point where we have

$$u_{bad}(x,t) = \int_{\mathbb{R}} f_{bad}(y) e^{-(x-y)^2/4t} (4\pi t)^{-1/2} dy = \int_{0}^{\infty} f(y) e^{-(x-y)^2/4t} (4\pi t)^{-1/2} dy.$$

The boundary condition demands that

$$u_{bad}(0,t) = \int_{0}^{\infty} f(y) e^{-y^2/(4t)} (4\pi t)^{-1/2} dy = 0.$$

Is that necessarily true? NO. For example, a perfectly decent initial data is the function

$$f(y) = \begin{cases} 1 & 2 < y < 4 \\ 0 & y \notin (2,4) \end{cases}.$$

For this function the solution found in this way would have

$$u_{bad}(0,t) = \int_2^4 e^{-y^2/(4t)} (4\pi t)^{-1/2} dy > 0.$$

On the other hand, the solution found in the oddly reflecting way would be

$$u_{good}(x,t) = \int_2^4 \left[e^{-(x-y)^2/(4t)} - e^{-(x+y)^2/(4t)} \right] (4\pi t)^{-1/2} dy.$$

This *does* satisfy the boundary condition.

1.1.2. *Returning to the good methods: in homogeneous heat equation on a half line.* Let us solve:

$$\partial_t u(x,t) - \partial_{xx} u(x,t) = G(x,t) \in \mathcal{L}^2, \quad x > 0, \quad t > 0,$$

with initial condition

$$u(x,0) = f(x) \in \mathcal{L}^2,$$

and boundary condition

$$u_x(0,t) = 0.$$

To deal with the inhomogeneity in the PDE we should divide this problem into two sub-problems. First we solve the homogeneous PDE with the initial condition. Then we will solve the inhomogeneous PDE with the initial condition equal to zero. The solution will be given by the sum of these two. In both cases we will keep the nice, homogeneous boundary condition.

So, to solve the homogeneous PDE, the boundary condition says that we should extend the initial data evenly. So, we let

$$f_e(-y) = f(y) \quad y > 0, \quad f_e(y) = f(y), \quad y > 0.$$

The heat kernel method gives rise to the solution

$$v(x,t) = \int_{\mathbb{R}} f_e(y) e^{-(x-y)^2/(4t)} (4\pi t)^{-1/2} dy.$$

Splitting the integral into the left and right sides, and using the definition of f_e, we have

$$v(x,t) = \int_{-\infty}^0 f(-y) e^{-(x-y)^2/(4t)} (4\pi t)^{-1/2} dy + \int_0^\infty f(y) e^{-(x-y)^2/(4t)} (4\pi t)^{-1/2} dy.$$

Making a substitution in the first integral above this becomes

$$v(x,t) = \int_0^\infty f(y) \left(e^{-(x+y)^2/(4t)} + e^{-(x-y)^2/(4t)} \right) (4\pi t)^{-1/2} dy.$$

Exercise 2. *Verify that this function satisfies the boundary condition at $x = 0$.*

Next we deal with the inhomogeneous PDE with the same nice boundary condition, but this time with the initial condition set to zero. We extend G evenly to all of $\mathbb{R}$ so that we can use the Fourier transform. So, we have

$$G_e(-y,t) := G(y,t) \quad y > 0, \quad G_e(y,t) = G(y,t), \quad y > 0.$$

We Fourier transform the heat equation in the x variable, obtaining the equation

$$\hat{w}_t(\xi,t) - \widehat{w_{xx}}(\xi,t) = \hat{G}_e(\xi,t).$$

Due to the lovely properties of the Fourier transform this becomes

$$\hat{w}_t(\xi,t) + \xi^2 \hat{w}(\xi,t) = \hat{G}_e(\xi,t).$$

This is an ordinary differential equation. The solution is given by the so-called mμthod (Kf):

$$\hat{w}(\xi,t) = \frac{\int_0^t \hat{G}_e(\xi,s) e^{\xi^2 s} ds + C}{e^{\xi^2 t}}.$$

To determine the constant, we use the initial condition. We would like $\hat{w}(\xi,0) = 0$. Hence $C = 0$. To make it more simple we put the denominator upstairs, so that

$$\hat{w}(\xi,t) = \int_0^t \hat{G}_e(\xi,s) e^{-\xi^2(t-s)} ds.$$

Inside the time integral we have the product of two Fourier transforms. Hence before Fourier transforming, this was a convolution. The function whose Fourier transform is $\hat{G}_e(\xi,s)$ is $G_e(x,s)$. The function whose Fourier transform is $e^{-\xi^2(t-s)}$ is $e^{-x^2/(t-s)}(4\pi(t-s))^{-1/2}$. Hence

$$w(x,t) = \int_0^t \int_{\mathbb{R}} G_e(y,s) e^{-(x-y)^2/(4(t-s))} (4\pi(t-s))^{-1/2} dyds.$$

Since G_e is even, a similar calculation shows that this gives

$$w(x,t) = \int_0^t \int_0^\infty G(y,s) \left(e^{-(x+y)^2/(4(t-s))} + e^{-(x-y)^2/(4(t-s))} \right) (4\pi(t-s))^{-1/2} dyds.$$

The full solution is then

$$u(x,t) = v(x,t) + w(x,t).$$

1.1.3. *Half line with unusual boundary condition.* Consider the equation:

$$\partial_t u - \partial_{xx} u = 0, \quad t, x > 0$$

$$u(x,0) = 0$$

$$u_x(0,t) = -c.$$

A natural thing to try to do is to deal with the inhomogeneous boundary condition using a steady state solution. Sadly, this will not work. A steady state solution should satisfy the homogeneous PDE, so we would need

$$-f''(x) = 0, \quad f'(0) = -c \implies f(x) = -cx + b.$$

Then we would solve the IVP for the homogeneous heat equation on the half line with the boundary condition $u_x(0,t) = 0$. There are two reasons this will not work.

(1) First, there is no way to determine what b should be.
(2) Second, no matter what the value of b, as long as $f(x)$ is not just the zero function, it is *not* in $\mathcal{L}^2$. Hence, we cannot Fourier transform and/or inverse Fourier transform. The results no longer hold true.

With the second consideration, we could try to use a different transform which does not require such strong decay properties. Recall that for the Laplace transform, we just need to have at most exponential growth, and define a function which is initially defined for a half-line to be zero on the negative half line. So, rather than Fourier transforming, we do Laplace transform here in the t variable. The PDE becomes

$$z\widetilde{u}(x,z) - \widetilde{u}(x,z) = 0.$$

This is because happily we have the condition $u(x,0) = 0$ so that term drops out. We can solve this ODE and obtain

$$\widetilde{u}(x,z) = a(z)e^{-x\sqrt{z}} + b(z)e^{x\sqrt{z}}.$$

The second term, since $x > 0$, will grow exponentially if the real part of z tends to ∞. That is not supposed to happen for Laplace transforms. So we try to solve using the first term only. To determine $a(z)$ we need the boundary condition. OBS! We must Laplace transform the boundary condition:

$$\widetilde{u}_x(0,z) = \mathfrak{L}(-c)(z).$$

We compute the Laplace transform of a constant to be:

$$\mathfrak{L}(-c)(z) = \int_0^\infty -ce^{-tz}dt = -c\frac{e^{-tz}}{-z}\Big|_0^\infty = -\frac{c}{z}.$$

We therefore need

$$\widetilde{u}_x(0,z) = -a(z)\sqrt{z} = -\frac{c}{z} \implies a(z) = \frac{c}{z\sqrt{z}}.$$

We have in this way found the Laplace transform of our solution:

$$\widetilde{u}(x,z) = \frac{c}{z\sqrt{z}}e^{-x\sqrt{z}}.$$

From this point in time we seek functions in the table who have the desired Laplace transform. The function

$$\Theta(t)\frac{e^{-x^2/(4t)}}{\sqrt{\pi t}} \text{ has Laplace transform } \frac{e^{-x\sqrt{z}}}{\sqrt{z}}.$$

We already saw that the function

$$c\Theta(t) \text{ has Laplace transform } \frac{c}{z}.$$

The Laplace transform takes convolutions to products. So, our solution is

$$u(x,t) = \int_{\mathbb{R}} c\Theta(t-s)\Theta(s)e^{-x^2/(4s)}(\pi s)^{-1/2}ds.$$

Due to the Heavyside functions, this is simply

$$u(x,t) = \int_0^t ce^{-x^2/(4s)}(\pi s)^{-1/2}ds.$$

1.2. **Fourier transform methods to compute impossible integrals.** We are given that a certain function has Fourier transform

$$\hat{f}(w) = \frac{w^2\Theta(w)}{(1+w^2)^2}.$$

We are asked to compute

$$\heartsuit = \int_{\mathbb{R}} f(t)e^{-|t|}\operatorname{sgn}(t)dt.$$

How on earth to do this? We use Plancharel's theorem which says that

$$\int_{\mathbb{R}} f(x)\overline{g(x)}dx = \frac{1}{2\pi}\int_{\mathbb{R}} \hat{f}(w)\overline{\hat{g}(w)}dw.$$

So, we need to compute the Fourier transform of $e^{-|t|}\operatorname{sgn}(t)$. Let's do this:

$$\hat{g}(w) = \int_{\mathbb{R}} e^{-|t|}\operatorname{sgn}(t)e^{-iwt}dt = \int_{-\infty}^{0} -e^{t(1-iw)}dt + \int_0^\infty e^{-t(1+iw)}dt$$
$$= -\frac{e^{t(1-iw)}}{1-iw}\bigg|_{-\infty}^{0} + \frac{e^{-t(1+iw)}}{-(1+iw)}\bigg|_0^\infty$$
$$= -\frac{1}{1-iw} + \frac{1}{1+iw} = \frac{-2iw}{1+w^2}.$$

So, we just need to compute

$$\frac{1}{2\pi}\int_{\mathbb{R}} \frac{w^2\Theta(w)}{(1+w^2)^2}\overline{\frac{(-2iw)}{1+w^2}}dw = \frac{i}{\pi}\int_0^\infty \frac{w^3}{(1+w^2)^3}dw.$$

We can handle this using integration by parts. Observe that

$$\left(\frac{1}{(1+w^2)^2}\right)' = -\frac{2(2w)}{(1+w^2)^3}.$$

So

$$\heartsuit = \frac{i}{\pi}\left(\frac{w^2}{(-4)(1+w^2)^2}\bigg|_0^\infty - \int_0^\infty \frac{2w}{(-4)(1+w^2)^2}dw\right)$$
$$= \frac{i}{2\pi}\int_0^\infty \frac{w}{(1+w^2)^2}dw = \frac{i}{2\pi}\ \frac{1}{(-2)(1+w^2)}\bigg|_0^\infty = \frac{i}{4\pi}.$$

How cute.

1.3. **Fourier transform methods to solve integral equations.** Upon popular demand, we should consider the equation

$$u(t) + \int_{-\infty}^{t} e^{\tau-t}u(\tau)d\tau = e^{-2|t|}.$$

The integral term looks very much like a convolution. Let us make it so. We need a function so that

$$f(t-\tau) = \begin{cases} e^{\tau-t} & \tau \le t \\ 0 & \tau > t \end{cases}.$$

The function

$$f(x) = \begin{cases} e^{x} & x \le 0 \\ 0 & x > 0 \end{cases}$$

does just this. So, our equation is now

$$u(t) + f * u(t) = e^{-2|t|}.$$

We apply the Fourier transform to the entire equation:

$$\hat{u}(\xi) + \hat{f}(\xi)\hat{u}(\xi) = \frac{4}{\xi^2 + 4}.$$

The right side of the equation was mercifully handed to us by a table. So we just need to compute the Fourier transform of f,

$$\hat{f}(\xi) = \int_{-\infty}^{0} e^x e^{-ix\xi} dx = \left. \frac{e^{x(1-i\xi)}}{1 - i\xi} \right|_{-\infty}^{0} = \frac{1}{1 - i\xi}.$$

So our equation becomes

$$\hat{u}(\xi) + \frac{\hat{u}(\xi)}{1 - i\xi} = \frac{4}{\xi^2 + 4}.$$

Solving for u we obtain

$$\hat{u}(\xi) = \frac{4(1 - i\xi)}{(\xi^2 + 4)(2 - i\xi)}.$$

The FIT says

$$u(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{ix\xi} \frac{4(1 - i\xi)}{(\xi^2 + 4)(2 - i\xi)} d\xi.$$

Fortunately for our studying-for-exam purposes, we do not need to compute such an integral. To do so would require the residue theorem from complex analysis, and we do not require you to do such calculations on the exam.

References

[1] Gerald B. Folland, *Fourier Analysis and Its Applications,* Pure and Applied Undergraduate Texts Volume 4, (1992).