FOURIER ANALYSIS & METHODS

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Abstract. Caveat Emptor! These are just informal lecture notes. Errors are inevitable! Read at your own risk! Also, this is by no means a substitute for the textbook, which is warmly recommended: *Fourier Analysis and Its Applications,* by Gerald B. Folland. He was the first math teacher I had at university, and he is awesome. A brilliant writer. So, why am I even doing this? Good question...

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Check mic, one two one two. Today we shall continue with examples.

1.1. **PDE on bounded interval with time dependent inhomogeneity in PDE.**

Consider the problem:

$$u_{tt} - u_{xx} = tf(x), \quad t > 0, \quad x \in [0,1],$$

subject to the boundary conditions:

$$u(0,t) = 5, \quad u(1,t) = 10,$$

and initial conditions:

$$u(x,0) = g(x), \quad u_t(x,0) = h(x).$$

The idea is to divide and conquer. First we deal with the most simple of these inhomogeneities, which are the boundary conditions. The procedure which (sound-checked) will work best is:

(1) use a steady-state solution to satisfy the boundary conditions and which vanishes when we apply the PDE
(2) find a solution to the homogeneous PDE which has homogeneous boundary conditions and satisfies the prescribed initial conditions *minus* the steady state solution found in the first step
(3) find a solution to the inhomogeneous PDE where all the other conditions (initial + boundary) are set to zero.

So we do this. We see first $v(x)$ which is independent of t and satisfies:

$$-v''(x) = 0, \quad v(0) = 5, \quad v(1) = 10.$$

The function v is therefore linear. The boundary conditions require:

$$v(x) = 5x + 5.$$

Next we seek a solution to the problem

$$w_{tt} - w_{xx} = 0, \quad w(0,t) = 0 = w(1,t),$$

with the initial conditions:

$$w(x,0) = g(x) - v(x), \quad w_t(x,0) = h(x).$$

The reason we don't need to clean up the steady state solution in the t derivative is because when we apply ∂_t to v it vanishes (since it doesn't depend on t). For this part, we can use our method of separation of variables! We write

$$w = XT \implies T''X - X''T = 0 \implies \frac{T''}{T} = \frac{X''}{X} = \text{ constant.}$$

We have the nice boundary conditions for X:

$$X(0) = X(1) = 0.$$

Exercise 1. *Show that the only solutions to*

$$X'' = \text{ constant times } X, \quad X(0) = X(1) = 0$$

are constant multiples of $X_n(x) = \sin(n\pi x)$. *Hint: consider the cases in which the constant is positive, zero, and negative separately.*

So, we have found the X_n and the constant:

$$X_n(x) = \sin(n\pi x), \quad \frac{X_n''}{X_n} = -n^2\pi^2.$$

This tells us that

$$\frac{T_n''}{T_n} = -n^2\pi^2 \implies T_n(t) = a_n \cos(n\pi t) + b_n \sin(n\pi t).$$

Since we are solving a homogeneous PDE, we can use superposition to smash all these together into a super solution:

$$w(x,t) = \sum_{n \geq 1} X_n(x) \left(a_n \cos(n\pi t) + b_n \sin(n\pi t) \right).$$

We use the IC to determine the constants. We wish for

$$w(x,0) = g(x) - v(x) = \sum_{n \geq 1} X_n(x) a_n \implies a_n = \frac{\int_0^1 (g(x) - v(x)) \overline{X_n(x)} dx}{\int_0^1 |X_n(x)|^2 dx}.$$

The other initial condition demands that

$$w_t(x,0) = h(x) = \sum_{n \geq 1} n\pi b_n X_n(x) \implies b_n = \frac{\int_0^1 h(x) \overline{X_n(x)} dx}{n\pi \int_0^1 |X_n(x)|^2 dx}.$$

Finally we deal with that inhomogeneous PDE, but set every other condition in the problem equal to zero. Hence we seek a solution to the problem:

$$\phi_{tt} - \phi_{xx} = t f(x), \quad t > 0, \quad x \in [0,1],$$

subject to the boundary conditions:

$$u(0,t) = 0, \quad u(1,t) = 0,$$

and initial conditions:

$$u(x,0) = 0, \quad u_t(x,0) = 0.$$

To solve this we use a series, writing

$$\phi(x,t) = \sum_{n \geq 1} X_n(x) c_n(t).$$

The functions X_n will guarantee that the nice homogeneous boundary conditions are satisfied. On the other side, we expand $tf(x)$ in a Fourier series with respect to the basis $\{X_n\}$. Here we note that the regular SLP theory guarantees that these are indeed a basis (we have already implicitly used this fact before when we solved for the coefficients in the function w above).

Let

$$\hat{f}_n := \frac{\int_0^1 f(x)\overline{X_n(x)}dx}{\int_0^1 |X_n(x)|^2 dx}.$$

Then we are solving the PDE:

$$\sum_{n\geq 1} c_n''(t)X_n(x) - c_n(t)X_n''(x) = \sum_{n\geq 1} t\hat{f}_n X_n(x).$$

Note that

$$X_n''(x) = -n^2\pi^2 X_n(x).$$

So our PDE is:

$$\sum_{n\geq 1} X_n(x)\left(c_n''(t) + n^2\pi^2 c_n(t)\right) = \sum_{n\geq 1} t\hat{f}_n X_n(x).$$

We equate the coefficients of $X_n(x)$ on both sides:

$$c_n''(t) + n^2\pi^2 c_n(t) = t\hat{f}_n.$$

This is a linear second order ODE. A particular solution is a linear function, namely

$$\frac{t\hat{f}_n}{n^2\pi^2}.$$

A solution to the homogeneous ODE is a linear combination of $\cos(n\pi t)$ and $\sin(n\pi t)$. Hence a general solution is of the form

$$c_n(t) = \alpha_n \cos(n\pi t) + \beta_n \sin(n\pi t) + \frac{t\hat{f}_n}{n^2\pi^2}.$$

To satisfy the zero initial conditions we wish that

$$c_n(0) = 0 \implies \alpha_n = 0.$$

We also wish for

$$c_n'(0) = 0 \implies n\pi\beta_n + \frac{\hat{f}_n}{n^2\pi^2} = 0 \implies \beta_n = -\frac{\hat{f}_n}{n^3\pi^3}.$$

Hence

$$c_n(t) = -\frac{\hat{f}_n}{n^3\pi^3}\sin(n\pi t) + \frac{t\hat{f}_n}{n^2\pi^2}.$$

The full solution is therefore given by

$$u(x,t) = \phi(x,t) + w(x,t) + v(x).$$

1.2. **PDE on bounded interval with time-dependent BC.** Next we wish to solve the problem:

$$u_t - u_{xx} = 0, \quad u(0,t) = t+1, \quad u(1,t) = 0, \quad u(x,0) = 1-x.$$

Well that's just dandy, the boundary condition depends on t.

Exercise 2. *Try to solve this equation using the Laplace transform in the t variable. (Hint: It's not going to be pretty nor is it going to work...) Still, it is good experience to do this in order to recognize the hallmarks of when one is going down the wrong path.*

Instead, let us seek to solve the boundary conditions by looking for a function of the form:

$$w(x,t) = f(t)g(x),$$

which satisfies

$$w(0,t) = 1+t, \quad w(1,t) = 0.$$

For the first condition, assuming $w(x,t) = f(t)g(x)$, we get that we need:

$$f(t)g(0) = 1+t \implies g(0) \text{ is constant, and } f(t) \text{ is equal to } (1+t)/g(0).$$

So, for the sake of simplicity, let us assume that

$$g(0) = 1 \implies f(t) = 1+t.$$

The second condition requires

$$w(1,t) = g(1)f(t) = g(1)(1+t) = 0 \implies g(1) = 0.$$

So, we have completely specified $f(t)$, and we have determined that we would like $g(0) = 1$ and $g(1) = 0$. We have not yet specified g. To see what g should be, let us turn to the PDE.

Ideally we would like the PDE to be zero, but let's see what happens when we apply the PDE to such a function:

$$f'(t)g(x) - g''(x)f(t) = g(x) - g''(x)(1+t).$$

The only way to get this to vanish, since g does not depend on t, is to demand that both $g(x)$ and $g''(x)$ vanish. This would ruin the condition that $g(0) = 1$. So we don't want that. The next best thing we can do is make the PDE as simple as possible. So, let's request that $g''(x) = 0$. Then g is a linear function. Since we wish for $g(0) = 1$ and $g(1) = 0$, the function which does this is

$$g(x) = 1 - x.$$

Hence

$$w(x,t) = (1-x)(1+t)$$

solves:

$$w(0,t) = 1+t, \quad w(1,t) = 0, \quad w_t - w_x x = 1 - x.$$

Moreover, we have

$$w(x,0) = 1 - x.$$

So, now we have something we can deal with using a steady state solution. In particular we seek a function v which only depends on x and which satisfies:

$$-v''(x) = -(1-x), \quad v(0) = 0 = v(1).$$

Then when we add them together, we get that $w + v$ satisfies:

$$w(0,t)+v(0) = 1+t, \quad w(1,t)+v(1) = 0, \quad w_t - w_{xx} - v_{xx} = 0, \quad w(x,0)+v(x) = 1-x+v(x).$$

So, we have the ODE for v, which is

$$v''(x) = x - 1, \quad v(0) = 0 = v(1) \implies v(x) = \frac{x^3}{6} - \frac{x^2}{2} + \left(\frac{1}{2} - \frac{1}{6}\right)x.$$

The final piece in our puzzle shall be solved by seeking a solution to

$$\phi_t - \phi_{xx} = 0, \quad \phi(0,t) = \phi(1,t) = 0, \quad \phi(x,0) = -v(x).$$

Then our solution will be

$$u(x,t) = w(x,t) + v(x) + \phi(x,t).$$

When we add everything up we get:

$$u(0,t) = w(0,t) + v(0) + \phi(0,t) = 1 + t,$$
$$u(1,t) = w(1,t) + v(1) + \phi(1,t) = 0,$$
$$u(x,0) = w(x,0) + v(x) + \phi(x,0) = 1 - x + v(x) - v(x) = 1 - x,$$
$$u_t - u_{xx} = w_t - w_{xx} - v''(x) = 1 - x - (1 - x) = 0.$$

So, let us find ϕ. Due to the nice boundary conditions, we can separate variables and recycle our previous calculations. To see how this works, put $\phi = TX$ into the PDE. It becomes

$$T'X - X''T = 0 \implies \frac{T'}{T} = \frac{X''}{X} \implies \text{ both sides are constant.}$$

Starting with the X side (why do we do this?) we are solving

$$X'' = \text{ constant times } X, \quad X(0) = X(1) = 0.$$

We have solved this very problem. We found that the only solutions are (up to constant factor)

$$X_n(x) = \sin(n\pi x), \quad X_n'' = -n^2\pi^2 X_n.$$

Consequently

$$\frac{T_n'}{T_n} = -n^2\pi^2 \implies T_n(t) = a_n e^{-n^2\pi^2 t}.$$

We use superposition since the PDE and everything (except the initial condition) is homogeneous to write

$$\phi(x,t) = \sum_{n \geq 1} X_n(x) a_n e^{-n^2\pi^2 t}.$$

The initial condition demands that

$$\phi(x,0) = \sum_{n \geq 1} X_n(x) a_n = -v(x) \implies a_n = \frac{\int_0^1 -v(x)\overline{X_n(x)}dx}{\int_0^1 |X_n(x)|^2 dx}.$$

1.3. **Computing mysterious sums.** Such an exercise is of the form: compute

$$\sum_{n \geq 0} \frac{1}{1+n^2}.$$

To make this feasible, we shall usually be given a hint. Here the hint would be to determine the Fourier series of e^x. So let us do this. The coefficients are

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^x e^{-inx} dx = \frac{1}{2\pi} \frac{e^{x(1-in)}}{1-in}\Big|_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi(1-in)}\left((-1)^n e^{\pi} - (-1)^n e^{-\pi}\right) = \frac{(-1)^n}{\pi(1-in)} \sinh(\pi).$$

The Fourier series is therefore:

$$\sum_{n \in \mathbb{Z}} c_n e^{inx} = \sum_{n \in \mathbb{Z}} \frac{(-1)^n \sinh(\pi)}{\pi(1-in)} e^{inx}.$$

There are two methods we can use. One method is to pick a clever choice of x and evaluate the series there.

1.3.1. *Method using pointwise convergence of Fourier series.*

Exercise 3. *Try using $x = 0$. See that it is not going to work.*

The reason that $x = 0$ will not work is due to that pesky alternating factor, $(-1)^n$. We won't be able to get rid of it. The sum we wish to compute does not have it. So, we simply won't be able to compute using $x = 0$. From whence did that factor of $(-1)^n$ come? This came from evaluating $e^{\pm in\pi}$. So, if we put in $x = \pm\pi$ then the series:

$$\sum_{n\in\mathbb{Z}} \frac{(-1)^n \sinh(\pi)}{\pi(1-in)} e^{\pm in\pi} = \sum_{n\in\mathbb{Z}} \frac{(-1)^n \sinh(\pi)}{\pi(1-in)} (-1)^n$$
$$= \sum_{n\in\mathbb{Z}} \frac{\sinh(\pi)}{\pi(1-in)} = \frac{\sinh(\pi)}{\pi} \sum_{n\in\mathbb{Z}} \frac{1}{1-in}.$$

That is because $(-1)^n(-1)^n = 1$ for all n. This is still not quite what we need. However we are closer. Notice that all integers, save zero, come in pairs. 1 and -1. 2 and -2. And so forth. Let us split up our series into the loner term and the rest of the pairs:

$$\frac{\sinh(\pi)}{\pi} \left(1 + \sum_{n\geq 1} \frac{1}{1-in} + \frac{1}{1+in} \right).$$

Something fantastic happens:

$$\frac{1}{1-in} + \frac{1}{1+in} = \frac{1+in+1-in}{1+n^2} = \frac{2}{1+n^2}.$$

So our series is:

$$\frac{\sinh(\pi)}{\pi} \left(1 + \sum_{n\geq 1} \frac{2}{1+n^2} \right).$$

To what does this series converge? The theorem says it converges to the average of the left and right limits of a certain function. Which function? It is the function which is equal to e^x in $(-\pi, \pi)$ and is defined on $\mathbb{R}$ to be 2π periodic, at the point π. Well, when we extend the function to be 2π periodic, it has jumps at the points π (and $-\pi$). Let's take the point π. Approaching it from the left, the function will tend to e^π. Approach the point π from the *right,* we are outside the interval where the function coincides with e^x. By 2π periodicity, for $x > \pi$, but close to π, the function is equal to $f(x-2\pi) = e^{x-2\pi}$. As $x \to \pi$ this goes to $e^{-\pi}$. Hence the Fourier series converges to

$$\frac{e^\pi + e^{-\pi}}{2} = \cosh(\pi).$$

We therefore have that our series sums to $\cosh(\pi)$. Hence:

$$\frac{\sinh(\pi)}{\pi} \left(1 + \sum_{n\geq 1} \frac{2}{1+n^2} \right) = \cosh(\pi).$$

We re-arrange, obtaining

$$\frac{\pi\cosh(\pi)}{\sinh(\pi)} = 1 + 2\sum_{n\geq 1} \frac{1}{1+n^2} \implies \frac{\pi\cosh(\pi)}{2\sinh(\pi)} - \frac{1}{2} = \sum_{n\geq 1} \frac{1}{1+n^2}.$$

This is not quite what we want, because the sum starts at one rather than zero. However, note that

$$\sum_{n\geq 0}\frac{1}{1+n^2} = 1+\sum_{n\geq 1}\frac{1}{1+n^2}.$$

So, we can easily fix this by adding one to both sides of our equality:

$$\frac{\pi\cosh(\pi)}{2\sinh(\pi)} - \frac{1}{2} + 1 = 1 + \sum_{n\geq 1}\frac{1}{1+n^2} = \sum_{n\geq 0}\frac{1}{1+n^2}.$$

1.3.2. *Method using Parseval's equality.* Parseval's equality may also be used. It says that

$$\sum_{n\in\mathbb{Z}} |c_n|^2 ||e^{inx}||^2 = \int_{-\pi}^{\pi} |e^x|^2 dx.$$

So, we just compute both sides:

$$|c_n|^2 = \frac{\sinh^2(\pi)}{\pi^2(1+n^2)}, \quad ||e^{inx}||^2 = 2\pi, \quad \int_{-\pi}^{\pi} |e^x|^2 dx = \left.\frac{e^{2x}}{2}\right|_{-\pi}^{\pi}$$

so

$$\sum_{n\in\mathbb{Z}} \frac{\sinh^2(\pi)}{\pi^2(1+n^2)} 2\pi = \left.\frac{e^{2x}}{2}\right|_{-\pi}^{\pi} = \sinh(2\pi).$$

Let us simplify and re-arrange a bit:

$$\frac{2\sinh^2(\pi)}{\pi}\sum_{n\in\mathbb{Z}}\frac{1}{(1+n^2)} = \sinh(2\pi) = 2\sinh(\pi)\cosh(\pi).$$

Above we used the double angle formula for the hyperbolic sine. This allows us to cancel the factors of $2\sinh(\pi)$ from both sides, obtaining

$$\frac{\sinh(\pi)}{\pi}\sum_{n\in\mathbb{Z}}\frac{1}{(1+n^2)} = \cosh(\pi).$$

Re-arranging we obtain:

$$\frac{\pi\cosh(\pi)}{\sinh(\pi)} = 1 + 2\sum_{n\geq 1}\frac{1}{1+n^2}.$$

Consequently,

$$\frac{\pi\cosh(\pi)}{2\sinh(\pi)} - \frac{1}{2} = \sum_{n\geq 1}\frac{1}{1+n^2}$$

so adding one to both sides again,

$$\frac{\pi\cosh(\pi)}{2\sinh(\pi)} - \frac{1}{2} + 1 = 1 + \sum_{n\geq 1}\frac{1}{1+n^2} = \sum_{n\geq 0}\frac{1}{1+n^2}.$$

1.4. **Per request a certain SLP.** This was skipped in lecture because it is not super relevant for the most pressing matter at hand... The problem was to solve:

$$u'' + \lambda u = 0, \quad u'(0) = u(0), \quad u(1) = 0.$$

Moreover, we should determine how many lambdas are in the interval $[-16, 16]$. We check for the different cases of λ first. Let us try $\lambda = 0$. This would give us a linear function. The condition that $u(1) = 0$ means the function is of the form $ax - a$, for some constant a. The condition $u'(0) = u(0)$ then requires $a = -a$. The only constant which satisfies this is $a = 0$. That means that $u = 0$ but the zero function is not an eigenfunction.

So let us proceed to checking positive lambdas. In this case we have

$$u(x) = a\cos(\sqrt{\lambda}x) + b\sin(\sqrt{\lambda}x).$$

The first boundary condition requires:

$$u'(0) = b\sqrt{\lambda} = u(0) = a \implies u(x) = b\left(\sqrt{\lambda}\cos(\sqrt{\lambda}x) + \sin(\sqrt{\lambda}x)\right).$$

The next boundary condition requires:

$$u(1) = b\left(\sqrt{\lambda}\cos(\sqrt{\lambda}) + \sin(\sqrt{\lambda})\right) = 0.$$

We do not wish for $b = 0$ because that would tear down our whole solution and make it all vanish. So we wish to find $\lambda > 0$ such that

$$\sqrt{\lambda} = -\tan(\sqrt{\lambda}).$$

Equivalently we look for solutions to the equation

$$-\mu = \tan(\mu), \quad \mu > 0, \quad \mu = \sqrt{\lambda}.$$

If we draw a picture, we see that the graphs of the functions $-\mu$ and $\tan(\mu)$ will intersect at zero, then they will intersect precisely once in the interval $(\pi/2, 3\pi/2)$, and again once in each interval of the form $((2n-1)\pi/2, (2n+1)\pi/2)$. Keep in mind that $\mu = \sqrt{\lambda}$.

Now let us check the case $\lambda < 0$. This will look almost the exact same, with just hyperbolic trig functions instead.

Exercise 4. *Show that the solutions in this case are of the form*

$$B\left(\sqrt{|\lambda|}\cosh(\sqrt{|\lambda|}x) + \sinh(\sqrt{|\lambda|}x)\right).$$

Show that the boundary condition at one requires that there exists a solution to

$$\tanh(\mu) = -\mu, \quad \mu = \sqrt{|\lambda|} > 0.$$

Show that for all $\mu > 0$ the function

$$\tanh(\mu) > 0.$$

Together with your teamwork computing the preceding exercise, we see that the case $\lambda < 0$ yields no solutions.

So we have found them all.

The question as to how many solutions are in the interval $[-16, 16]$ can be answered by considering a picture. First, there are no solutions in $[-16, 0]$. The first positive solution occurs in the interval $\mu = \sqrt{\lambda} \in (\pi/2, 3\pi/2)$. The next positive solution occurs in the interval $(3\pi/2, 5\pi/2)$. Note that this corresponds to the square root of lambda, and that

$$\frac{3\pi}{2} > \frac{9}{2} > 4.$$

So, if $\sqrt{\lambda} \in (3\pi/2, 5\pi/2)$ then $\lambda > 16$. However, the solution in the interval $(\pi/2, 3\pi/2)$ occurs in the part where the tangent is negative, and that is between $(\pi/2, \pi)$. When the square root of lambda is in here, then $\sqrt{\lambda} < \pi \implies \lambda < \pi^2 < 16$. So there is precisely one solution λ in the interval $[-16, 16]$.