FOURIER ANALYSIS & METHODS

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Abstract. Caveat Emptor! These are just informal lecture notes. Errors are inevitable! Read at your own risk! Also, this is by no means a substitute for the textbook, which is warmly recommended: *Fourier Analysis and Its Applications,* by Gerald B. Folland. He was the first math teacher I had at university, and he is awesome. A brilliant writer. So, why am I even doing this? Good question...

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Proposition 1. *On the interval $[-\pi, \pi]$, the functions*

$$\phi_n(x) = \frac{e^{inx}}{\sqrt{2\pi}}$$

are an orthonormal set with respect to the scalar product,

$$\langle f, g \rangle = \int_{-\pi}^{\pi} f(x)\overline{g(x)}dx.$$

Proof: First, we show that these guys are orthogonal. To do that, we just take $m \neq n$ and compute

$$\int_{-\pi}^{\pi} e^{inx}\overline{e^{imx}}dx.$$

Of course, the 2π factors don't matter. They're not going to make the inner product vanish! We recall that

$$\overline{e^{imx}} = e^{-imx}.$$

Exercise 1. *Why is this true? Explain in your own words or prove it algebraically.*

So, we compute,

$$\int_{-\pi}^{\pi} e^{ix(n-m)}dx = 2\pi \text{ m=n}, \int_{-\pi}^{\pi} e^{ix(n-m)}dx = \left.\frac{e^{ix(n-m)}}{n-m}\right|_{x=-\pi}^{\pi} n \neq m.$$

Now, I claim that the function $e^{ix(n-m)}$ is 2π-periodic. We compute

$$e^{i(x+2\pi)(n-m)} = e^{ix(n-m)}e^{2\pi i(n-m)}.$$

Since $n - m \in \mathbb{Z}$, $e^{2\pi i(n-m)} = 1$. Consequently,

$$\frac{e^{i\pi(n-m)}}{n-m} = e^{-i\pi(n-m)}n - m,$$

so

$$\left.\frac{e^{ix(n-m)}}{n-m}\right|_{x=-\pi}^{\pi} = 0, \quad n \neq m.$$

Consequently, we have proven that

$$\langle \phi_n, \phi_m \rangle = 0, \quad n \neq m.$$

We also computed

$$\langle \phi_n, \phi_n \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{inx} e^{-inx} dx = 1.$$

This is precisely what it means to be orthonormal.

So, now we know that $\{\phi_n(x)\}_{n\in\mathbb{Z}}$ are an orthonormal *set.* We want them to actually be an orthonormal *basis,* so that we can write for any $u_0(x)$,

$$u_0(x) = \sum_{n\in\mathbb{Z}} c_n \phi_n(x), \quad \phi_n(x) = \frac{e^{inx}}{\sqrt{2\pi}}.$$

In analogue to linear algebra, we should expect the coefficients to be the scalar product of our function $u_0(x)$ with the basis functions (vectors), $\phi_n(x)$. More generally, for a 2π periodic function $v(x)$, we hope to be able to write it as

$$v(x) = \sum_{n\in\mathbb{Z}} c_n \phi_n(x), \quad c_n = \int_{-\pi}^{\pi} v(x)\overline{\phi_n(x)} dx = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} v(x) e^{-inx} dx,$$

so that

$$v(x) = \sum_{n\in\mathbb{Z}} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} f(\xi) e^{-in\xi} d\xi \right) e^{inx}.$$

This motivates:

Definition 2. Assume f is periodic on $[-\pi, \pi]$ with period 2π. Define

$$c_n := \frac{1}{2\pi} \langle f, e^{inx} \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx.$$

These are the Fourier coefficients of f. The Fourier series of f is

$$\sum_{n\in\mathbb{Z}} c_n e^{inx}.$$

So, the real question is, *when does the Fourier series actually converge to equal* $f(x)$*?*

Exercise 2. *If f is as in the definition and is also even, prove that $b_n = 0$ for all n. If f is as in the definition and is also odd, prove that $a_n = 0$ for all n. (Hint: If you forgot what a_n and b_n are, look at the previous exercise!).*

1.0.1. *Examples.* Consider the function $f(x) = |x|$. It satisfies $f(-\pi) = f(\pi)$. We can just *make* it 2π-periodic by extending it to $\mathbb{R}$ to satisfy $f(x + 2\pi) = f(x)$ for all x. The graph then looks like a zig-zag or sawtooth. We compute the Fourier coefficients:

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} |x| e^{-inx} dx, \quad c_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |x| dx = \frac{2\pi^2}{2(2\pi)} = \frac{\pi}{2}.$$

Since

$$|x| = \begin{cases} -x & x < 0 \\ x & x \geq 0 \end{cases}$$

we compute:

$$\int_{-\pi}^{0} -x e^{-inx} dx, \quad \int_{0}^{\pi} x e^{-inx} dx.$$

We do substitution in the first integral to change it:

$$\int_{-\pi}^{0} -xe^{-inx} dx = \int_{0}^{\pi} xe^{inx} dx = \frac{xe^{inx}}{in}\bigg|_0^\pi - \int_0^\pi \frac{e^{inx}}{in} dx$$

$$= \frac{\pi e^{in\pi}}{in} - \frac{e^{in\pi}}{(in)^2} + \frac{1}{(in)^2}.$$

Similarly we also use integration by parts to compute

$$\int_0^\pi xe^{-inx} dx = \frac{xe^{-inx}}{-in}\bigg|_0^\pi - \int_0^\pi \frac{e^{-inx}}{(-in)} dx$$

$$= \frac{\pi e^{-in\pi}}{-in} - \frac{e^{-in\pi}}{(-in)^2} + \frac{1}{(-in)^2}.$$

Adding them up and using the 2π periodicity, we get

$$\frac{2e^{in\pi}}{n^2} - \frac{2}{n^2} = \frac{2(-1)^n - 2}{n^2}.$$

OBS! We need to divide by 2π to get

$$c_n = \frac{(-1)^n - 1}{\pi n^2}, \quad n \in \mathbb{Z} \setminus \{0\}.$$

The Fourier series is therefore

$$\frac{\pi}{2} + \sum_{n \in \mathbb{Z},\ \text{odd}} e^{inx} \left(-\frac{2}{\pi n^2} \right).$$

Exercise 3. *Use these calculations to compute the Fourier cosine series, that is the series*

$$\sum_{n \geq 0} a_n \cos(nx).$$

Exercise 4. *Next, consider the function $f(x) = x$ initially on the interval $]-\pi, \pi[$. We extend it in a similar way to be 2π periodic, but it will then be discontinuous with jump discontinuities at odd-integer multiples of π. Compute in the same way the Fourier coefficients of this function, that is, compute*

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} xe^{-inx} dx \quad n \in \mathbb{Z}.$$

Use that calculation to show that $a_n = 0$ for all n, and then compute the Fourier sine series,

$$\sum_{n \geq 1} b_n \sin(nx).$$

Exercise 5. *Look at these two examples. Do the series converge? Do they converge absolutely? Compare and contrast them!*

1.1. **Introducing Hilbert spaces.** A Hilbert space is a complete normed vector space whose norm is induced by a scalar product.

Definition 3. A Hilbert space, H, is a vector space. This means that H is a set which contains elements. If f and g are elements of H, then for any $a, b \in \mathbb{C}$ we have

$$af + bg \in H.$$

This is what it means to be a vector space. Moreover, Hilbert spaces have two other nice features: a scalar product and a norm. Let us write the scalar product as

$$\langle f, g \rangle : H \times H \to \mathbb{C}.$$

To be a scalar product it must satisfy:

$$\langle af, g \rangle = a \langle f, g \rangle \quad \forall a \in \mathbb{C},$$

$$\langle h + f, g \rangle = \langle h, g \rangle + \langle f, g \rangle,$$

and

$$\langle f, g \rangle = \overline{\langle g, f \rangle}.$$

The norm is defined through the scalar product via:

$$||f|| := \sqrt{\langle f, f \rangle}.$$

Finally, what it means to be complete is that if a sequence $\{f_n\} \in H$ is Cauchy, which means that for any $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$||f_n - f_m|| < \varepsilon \quad \forall n, m \geq N,$$

then there exists $f \in H$ such that

$$\lim_{n \to \infty} f_n = f,$$

by which we mean that

$$\lim_{n \to \infty} ||f_n - f|| = 0.$$

Exercise 6. *As an example, we can take $H = \mathbb{C}^n$. For $z = (z_1, z_2, \ldots, z_n) \in \mathbb{C}^n$ and $w = (w_1, \ldots, w_n) \in \mathbb{C}^n$ the scalar product*

$$\langle z, w \rangle := \sum_{j=1}^{n} z_j \overline{w_j}.$$

Show that the scalar product defined in this way satisfies all the demands made upon it in the definition above. Why is $H = \mathbb{C}^n$ complete?

Now, let us fix a finite (not infinite) interval $[a, b]$. We shall be particularly interested in a Hilbert space known as $L^2([a, b])$ or once we have specified a and b, simply L^2.

Definition 4 (The real one)**.** $L^2([a, b])$ is the set of equivalence of classes of functions where f and g are equivalent if

$f(x) = g(x)$ for almost every $x \in [a, b]$ with respect to the one dimensional Lebesgue measure.

Moreover, for any f belonging to such an equivalence class, we require

l2finite (1.1) $$\int_a^b |f(x)|^2 dx < \infty.$$

If f and g are each members of equivalence classes satisfying (1.1) the scalar product of f and g is then defined to be

$$\langle f, g \rangle = \int_a^b f(x)\overline{g(x)}dx. \tag{1.2}$$

One can prove that with this definition we obtain a Hilbert space.

Theorem 5. *The space $L^2([a,b])$ for any bounded interval $[a,b]$ defined as above, with the scalar product defined as above, is a Hilbert space.*

This theorem is beyond the scope of this course. Moreover, the "real definition" of L^2 is also a bit much. This is why I offer you:

Definition 6 (The workable one)**.** $L^2([a,b])$ is the set of functions which satisfy (1.1), and is equipped with the scalar product defined in (1.2).

Although we don't necessarily need it right now, you may be happy to know that the L^2 scalar product satisfies a Cauchy-Schwarz inequality,

$$|\langle f, g \rangle| \leq ||f||||g||.$$

Exercise 7. *Use the Cauchy-Schwarz inequality to prove that for any $f \in L^2$ on the interval $[-\pi, \pi]$, the Fourier coefficients,*

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx,$$

satisfy

$$|c_n| \leq \frac{||f||}{\sqrt{2\pi}}.$$