FOURIER ANALYSIS & METHODS

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ABSTRACT. Caveat Emptor! These are just informal lecture notes. Errors are inevitable! Read at your own risk! Also, this is by no means a substitute for the textbook, which is warmly recommended: *Fourier Analysis and Its Applications,* by Gerald B. Folland. He was the first math teacher I had at university, and he is awesome. A brilliant writer. So, why am I even doing this? Good question...

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What are some examples? Well, any function f which is bounded on the interval will be an L^2 function. Let's make this official in what we'll call the standard estimate.

Proposition 1 (The standard estimate)**.** *Assume f is defined on some interval $[a, b]$. Assume that f satisfies a bound of the form $|f(x)| \le M$ for $x \in [a, b]$.*[1] *Then,*

$$\left| \int_a^b f(x) dx \right| \le (b-a)M.$$

Proof: Standard estimate!

$$\left| \int_a^b f(x) dx \right| \le \int_a^b |f(x)| dx \le \int_a^b M dx = M(b-a).$$

So, if f is bounded on an interval, then $|f|^2 \le M^2$, is also bounded, hence the integral is bounded. Something like $f(x) = \frac{1}{x}$ will be problematic if the interval contains 0. However, even though $f(x) = x^{-1/3}$ blows up as $x \to 0$, it blows up slowly enough that

$$\int_{-\pi}^{\pi} |x^{-1/3}|^2 dx < \infty.$$

So, the function doesn't have to be bounded for the integral to be finite, but it also can't blow up too badly.

2. Bessel's Inequality (L^2 convergence of Fourier series)

Today we're going to investigate the issue of convergence of Fourier series. To move towards this question of convergence, we prove an important estimate known as the Bessel Inequality.

[1] We actually only need this for "almost every" x, but to make that precise, we need some Lebesgue measure theory.

Theorem 2 (Bessel Inequality)**.** *Assume that f is 2π periodic and integrable on $[-\pi, \pi]$. Then the Fourier coefficients $\{c_n\}_{n\in\mathbb{Z}}$ satisfy*

$$\sum_{n\in\mathbb{Z}} |c_n|^2 \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2 dx.$$

Proof: It is sufficient to show that

$$2\pi \sum_{n=-N}^{N} |c_n|^2 \leq ||f||^2 \quad \forall N \in \mathbb{N}.$$

Since on the right side we have the L^2 norm of a function, we would like to have the L^2 norm of a function. Recall the Pythagorean Theorem: when $a \perp b$ then the length of the vector $a + b = c$ is equal to $a^2 + b^2$. The same thing works in higher dimensions. In particular, since the functions e^{inx} are orthogonal for $n \neq m$, it is also true that $c_n e^{inx}$ are orthogonal for $n \neq m$, so we have

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$$||\sum_{n=-N}^{N} c_n e^{inx}||^2 = \sum_{n=-N}^{N} ||c_n e^{inx}||^2 = \sum_{n=-N}^{N} 2\pi |c_n|^2. \tag{2.1}$$

Now, let's write

$$S_N(x) := \sum_{n=-N}^{N} c_n e^{inx}.$$

This is the partial Fourier expansion of f. Let us compare it to f using the L^2 norm:

$$0 \leq ||S_N - f||^2 = \langle S_N - f, S_N - f\rangle = \langle S_N, S_N - f\rangle - \langle f, S_N - f\rangle$$
$$= \langle S_N, S_N\rangle - \langle S_N, f\rangle - \langle f, S_N\rangle + \langle f, f\rangle$$
$$= ||S_N||^2 - \langle S_N, f\rangle - \langle f, S_N\rangle + ||f||^2.$$

Let us compute the two terms in the middle:

$$\langle S_N, f\rangle = \int_{-\pi}^{\pi} \sum_{n=-N}^{N} c_n e^{inx}\overline{f(x)} dx = \sum_{n=-N}^{N} c_n \int_{-\pi}^{\pi} e^{inx}\overline{f(x)} dx = \sum_{n=-N}^{N} c_n \overline{\int_{-\pi}^{\pi} e^{-inx} f(x) dx}$$
$$= \sum_{n=-N}^{N} c_n 2\pi \overline{c_n}.$$

We compute:

$$\langle f, S_N\rangle = \int_{-\pi}^{\pi} f(x) \sum_{n=-N}^{N} \overline{c_n e^{inx}} dx = \sum_{n=-N}^{N} \overline{c_n} \int_{-\pi}^{\pi} f(x) e^{-inx} dx = \sum_{n=-N}^{N} \overline{c_n} 2\pi c_n.$$

Since

$$|c_n|^2 = c_n \overline{c_n}$$

we have

$$0 \leq ||S_N - f||^2 = ||S_N||^2 - \langle S_N, f\rangle - \langle f, S_N\rangle + ||f||^2 = ||S_N||^2 - 2(2\pi) \sum_{n=-N}^{N} |c_n|^2 + ||f||^2.$$

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By (2.1), we have

$$0 \leq 2\pi \sum_{n=-N}^{N} |c_n|^2 - 2(2\pi) \sum_{n=-N}^{N} |c_n|^2 + ||f||^2 \implies 2\pi \sum_{n=-N}^{N} |c_n|^2 \leq ||f||^2.$$

Corollary 3. *We have*

$$\sum_{n\in\mathbb{N}} |a_n|^2 + |b_n|^2 = 4|c_0|^2 + 2 \sum_{n\in\mathbb{Z}\setminus 0} |c_n|^2,$$

and

$$\lim_{|n|\to\infty} \star_n = 0, \quad \star = a, b, \text{ or } c.$$

Exercise 1. *The proof is an exercise. First, use the previous exercises to express the a's and b's in terms of the c's. Next, what can you say about the terms of a non-negative, convergent series?*

2.1. **Pointwise convergence of Fourier Series.** By Bessel's inequality, we know that

$$\sum_{n\in\mathbb{Z}} |c_n|^2 \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |f|^2.$$

Now, it's important to note that when the series of $|c_n|^2$ converges, then eventually $|c_n|^2 < 1$ so also $|c_n| < 1$. Then, $|c_n| > |c_n|^2$. So, just because the series of $|c_n|^2$ converges, the series with just c_n might not. For example,

$$\sum_{n\geq 1} \frac{1}{n^2} < \infty$$

whereas

$$\sum_{n\geq 1} \frac{1}{n} = \infty.$$

So Bessel's inequality doesn't tell us that the Fourier series

$$\sum_{n\in\mathbb{Z}} c_n e^{inx}$$

always converges. This is a bit of a concern, because we want to use our method to solve PDEs. If our solution is one of these Fourier series, then we're up a creek without a paddle if that series doesn't converge to anything! This is the motivation to investigate the subtle question of pointwise convergence of Fourier series. Although math is fun just for itself, here, we're always motivated by a desire to understand real, relevant, physical and chemical processes! (Like heat, waves, electromagnetism, quantum particles, chemical reactions, the hydrogen and other atoms, etc...)

Definition 4. A function is piecewise $\mathcal{C}^k$ on a (possibly infinite) interval, I, if there is a discrete set, S of points in the interval (possibly empty set) such that f is $\mathcal{C}^k$ on $I \setminus S$. Moreover, we assume that the left and right limits of $f^{(j)}$ exist at all of the points in S, for all $j = 0, 1, \ldots, k$.

In case the notion of discrete set is unfamiliar, if the set S contains finitely many points, then it's a discrete set. If the interval $I = \mathbb{R}$, then both $\mathbb{Z}$ and $\mathbb{N}$ are discrete sets, but $\mathbb{Q}$ is not. Any discrete set in $\mathbb{R}$ is countable, so we may write such a set as $\{p_n\}_{n\in\mathbb{N}}$. Moreover, to be discrete, for each p_n there exists $\varepsilon_n > 0$ such that $|p_n - p_m| > \varepsilon_n \forall m \neq n$. That is, in the little interval $[p_n - \varepsilon_n/2, p_n + \varepsilon_n/2]$, the only point of our discrete set contained in that interval is p_n.

Examples of piecewise $\mathcal{C}^1$ functions are our periodically extended $|x|$, which is continuous on $\mathbb{R}$ but only piecewise $\mathcal{C}^1$. The periodically extended x is piecewise $\mathcal{C}^0$ and also piecewise $\mathcal{C}^1$. Actually, both of these guys are piecewise $\mathcal{C}^\infty$, because apart from the odd multiples of π, (and 0 for $|x|$) these functions are lovely and smooth.

Now we are going to prove the great big theorem about pointwise convergence of Fourier series.

Theorem 5. *Let f be a 2π periodic function. Assume that f is piecewise $\mathcal{C}^1$ on $\mathbb{R}$, where piecewise $\mathcal{C}^1$ is defined as above. Denote the left limit at x by $f(x_-)$ and the right limit by $f(x_+)$. Let*

$$S_N(x) = \sum_{-N}^{N} c_n e^{inx},$$

where

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx.$$

Then

$$\lim_{N\to\infty} S_N(x) = \frac{1}{2}\left(f(x_-) + f(x_+)\right), \quad \forall x \in \mathbb{R}.$$

Proof: This is a big theorem, because it's got a lot of clever ideas in the proof. Smaller theorems can be proven by just "following your nose." So, to try to help with the proof, we're going to highlight the big ideas. To learn the proof, you can start by learning all the big ideas in the order in which they're used. Once you've got these down, then try to fill in the math steps starting at one idea, working to get to the next idea. The big ideas are like light posts guiding your way through the dark and spooky math.

Idea 1: Fix a point $x \in \mathbb{R}$. We want to prove that

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$$\lim_{N\to\infty} S_N(x) = \frac{1}{2}\left(f(x_-) + f(x_+)\right). \tag{2.2}$$

Idea 2: Expand the series $S_N(x)$ using its definition.

$$S_N(x) = \sum_{-N}^{N} \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)e^{-iny}dye^{inx}.$$

Now, let's move that lonely e^{inx} inside the integral so it can get close to its friend, e^{-iny}. Then,

$$S_N(x) = \sum_{-N}^{N} \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)e^{-iny+inx}dy.$$

We want to prove (fseriesconvg 2.2). Above we have $f(y)$ rather than $f(x)$. This leads us to...

Idea 3: Change the variable. Let $t = y - x$.

Then $y = t + x$. We have

$$S_N(x) = \sum_{-N}^{N} \frac{1}{2\pi} \int_{-\pi-x}^{\pi-x} f(t+x)e^{-int}dt.$$

Remember that very first fact we proved for periodic functions? It said that the integral of a periodic function of period P from any point a to $a+P$ is the same, no matter what a is. Here $P = 2\pi$. This leads to...

Idea 4: Use the Lemma on integrals of periodic functions to shift the integral

$$\int_{-\pi-x}^{\pi-x} f(t+x)e^{-int}dt = \int_{-\pi}^{\pi} f(t+x)e^{-int}dt.$$

Thus

$$S_N(x) = \sum_{-N}^{N} \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t+x)e^{-int}dt = \int_{-\pi}^{\pi} f(t+x) \frac{1}{2\pi} \sum_{-N}^{N} e^{int}dt.$$

Idea 4: Define the N^{th} Dirichlet kernel, $D_N(t)$.

$$D_N(t) = \frac{1}{2\pi} \sum_{-N}^{N} e^{int}.$$

Idea 5: Collect the even and odd terms of D_N to compute its integral.

Recall that

$$n \in \mathbb{N} \implies e^{int} + e^{-int} = 2\cos(nt), n > 0.$$

Hence, we can pair up all the terms ± 1, ± 2, etc, and write

$$D_N(t) = \frac{1}{2\pi} + \sum_{n=1}^{N} \frac{1}{\pi} \cos(nt).$$

So, $D_N(t)$ is an even function. Moreover, since $\cos(nt)$ is 2π periodic and even,

$$\int_{-\pi}^{\pi} \cos(nt)dt = 0 \quad \forall n \geq 1,$$

so

$$\int_{-\pi}^{\pi} D_N(t)dt = \int_{-\pi}^{\pi} \frac{1}{2\pi} dt = 1.$$

Since $D_N(t)$ is even, we also have:

dnint (2.3)
$$\int_{-\pi}^{0} D_N(t)dt = \frac{1}{2} = \int_{0}^{\pi} D_N(t)dt.$$

Idea 6: Go back to the original definition of $D_N(t)$ and re-write it to look like a geometric series.

As it stands, $D_N(t)$ looks almost like a geometric series, but the problem is that it goes from minus exponents to positive ones. We can fix that right up by factoring out the largest negative exponent, so

$$D_N(t) = \frac{1}{2\pi} e^{-iNt} \sum_{n=0}^{2N} e^{int}.$$

We know how to sum a partial geometric series, don't we? This gives

dngeo (2.4)
$$D_N(t) = \frac{1}{2\pi} e^{-iNt} \frac{1 - e^{i(2N+1)t}}{1 - e^{it}} = \frac{e^{-iNt} - e^{i(N+1)t}}{2\pi(1 - e^{it})}.$$

Since

$$S_N(x) = \int_{-\pi}^{\pi} f(t+x) D_N(t) dt,$$

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(2.2) is equivalent to

$$\lim_{N\to\infty} \left| \int_{-\pi}^{\pi} f(t+x) D_N(t) dt - \frac{1}{2}\left(f(x_-) + f(x_+)\right) \right| = 0.$$

The S_N business has an integral, but the $f(x_\pm)$ don't. They have got a convenient factor of one half, so...

Idea 7: Use our calculation of the integral of D_N to write

$$\frac{1}{2} f(x_-) = \int_{-\pi}^{0} D_N(t) dt f(x_-), \quad \frac{1}{2} f(x_+) = \int_{0}^{\pi} D_N(t) dt f(x_+).$$

Hence we are bound to prove that

$$\lim_{N\to\infty} \left| \int_{-\pi}^{\pi} f(t+x) D_N(t) dt - \int_{-\pi}^{0} D_N(t) f(x_-) dt - \int_{0}^{\pi} D_N(t) f(x_+) dt \right| = 0.$$

It is quite natural now to split the integral into the left and right sides, so that we must prove

$$\lim_{N\to\infty} \left| \int_{-\pi}^{0} D_N(t)(f(t+x) - f(x_-)) dt + \int_{0}^{\pi} D_N(t)(f(t+x) - f(x_+)) dt \right|.$$

Idea 8: Use the second property (2.4) dngeo we proved about $D_N(t)$.

$$\left| \int_{-\pi}^{0} D_N(t)(f(t+x) - f(x_-)) dt + \int_{0}^{\pi} D_N(t)(f(t+x) - f(x_+)) dt \right| =$$

$$\left| \int_{-\pi}^{0} \frac{e^{-iNt} - e^{i(N+1)t}}{2\pi(1-e^{it})}(f(t+x) - f(x_-)) dt + \int_{0}^{\pi} \frac{e^{-iNt} - e^{i(N+1)t}}{2\pi(1-e^{it})}(f(t+x) - f(x_+)) dt \right|.$$

Since there are these factors of e^{-iNt} and $e^{i(N+1)t}$, this sort of looks like some twisted version of a Fourier coefficient. This observation leads us to...

Idea 9: Define a new function

$$g(t) = \frac{f(t+x) - f(x_-)}{1 - e^{it}}, \quad \text{for } t \in [-\pi, 0),$$

$$g(t) = \frac{f(t+x) - f(x_+)}{1 - e^{it}}, \quad \text{for } t \in (0, \pi].$$

The function g is well-defined on the interval $[-\pi, \pi] \setminus \{0\}$ because the denominator does not vanish there. Moreover, it has the same properties as f has on this interval. We extend g to all of $\mathbb{R}$ to be 2π periodic. What happens to g when $t \to 0$?

$$\lim_{t\to 0_-} \frac{f(t+x) - f(x_-)}{1 - e^{it}} = \lim_{t\to 0_-} \frac{t(f(t+x) - f(x_-))}{t(1 - e^{it})} = \frac{f'(x_-)}{-ie^{i0}} = if'(x_-).$$

For the other side, a similar argument shows that

$$\lim_{t\to 0_+} \frac{f(t+x) - f(x_-)}{1 - e^{it}} = if'(x_+).$$

Therefore, g has finite left and right limits at $t = 0$, because f does. Hence, g is also a piecewise differentiable and piecewise continuous 2π periodic function. Consequently, g is in particular bounded on $[-\pi, \pi]$ so it is in $L^2([-\pi, \pi])$ and Bessel's inequality holds.

Idea 10: Recognize the Fourier coefficients of the new function

$$\int_{-\pi}^{0} \frac{e^{-iNt} - e^{i(N+1)t}}{2\pi(1-e^{it})}(f(t+x)-f(x_-))dt + \int_{0}^{\pi} \frac{e^{-iNt} - e^{i(N+1)t}}{2\pi(1-e^{it})}(f(t+x)-f(x_+))dt$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-iNt}g(t)dt - \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(N+1)t}g(t)dt.$$

The first term above is by definition G_N, the N^{th} Fourier coefficient of g, whereas the second term above is by definition G_{-N-1}, the $-N-1$ Fourier coefficient of g. By Bessel's inequality,

$$\lim_{N\to\infty} G_N = 0 = \lim_{N\to\infty} G_{-N-1}.$$

2.2. **Example of using Fourier series to compute sums.** We wish to compute:

$$\sum_{n=0}^{\infty} \frac{1}{1+n^2}.$$

Hint: Expand e^x in a Fourier series on $(-\pi,\pi)$). Often, you'll be given such a hint, as when this problem appeared on an exam...

We follow the hint. We need to compute

$$\int_{-\pi}^{\pi} e^x e^{-inx}dx = \frac{e^{x(1-in)}}{1-in}\Big|_{x=-\pi}^{x=\pi} = \frac{e^{\pi}e^{-in\pi}}{1-in} - \frac{e^{-\pi}e^{in\pi}}{1-in} = (-1)^n \frac{2\sinh(\pi)}{1-in}.$$

Hence, the Fourier coefficients are

$$\frac{1}{2\pi}(-1)^n\frac{2\sinh(\pi)}{1-in},$$

and the Fourier series for e^x on this interval is

$$e^x = \sum_{-\infty}^{\infty} \frac{(-1)^n \sinh(\pi)}{\pi(1-in)}e^{inx}, \quad x \in (-\pi,\pi).$$

We can pull out some constant stuff,

$$e^x = \frac{\sinh(\pi)}{\pi}\sum_{-\infty}^{\infty} \frac{(-1)^n e^{inx}}{1-in}, \quad x \in (-\pi,\pi).$$

Now, we use the theorem which tells us that the series converges to the average of the left and right hand limits at points of discontinuity, like for example π. The left limit is e^{π}. Extending the function to be 2π periodic, means that the right limit approaching π is equal to $e^{-\pi}$. Hence

$$\frac{e^{\pi}+e^{-\pi}}{2} = \frac{\sinh(\pi)}{\pi}\sum_{-\infty}^{\infty}\frac{(-1)^n e^{in\pi}}{1-in}.$$

Now, we know that $e^{in\pi} = (-1)^n$, thus

$$\frac{e^{\pi}+e^{-\pi}}{2} = \frac{\sinh(\pi)}{\pi}\sum_{-\infty}^{\infty}\frac{1}{1-in}.$$

We now consider the sum, and we pair together $\pm n$ for $n \in \mathbb{N}$, writing

$$\sum_{-\infty}^{\infty} \frac{1}{1-in} = 1 + \sum_{n\in\mathbb{N}} \frac{1}{1-in} + \frac{1}{1+in} = 1 + \sum_{n\in\mathbb{N}} \frac{2}{1+n^2}.$$

Hence we have found that

$$\frac{e^{\pi}+e^{-\pi}}{2} = \frac{\sinh(\pi)}{\pi} \sum_{-\infty}^{\infty} \frac{(-1)^n e^{in\pi}}{1-in} = \frac{\sinh(\pi)}{\pi} \left(1 + \sum_{n\in\mathbb{N}} \frac{2}{1+n^2} \right).$$

The rest is mere algebra. On the left we have the definition of $\cosh(\pi)$. So, moving over the $\sinh(\pi)$ we have

$$\frac{\pi\cosh(\pi)}{\sinh(\pi)} = 1 + 2\sum_{n\in\mathbb{N}} \frac{1}{1+n^2} \implies \left(\frac{\pi\cosh(\pi)}{\sinh(\pi)} - 1 \right) \frac{1}{2} = \sum_{n\in\mathbb{N}} \frac{1}{1+n^2}.$$

Wow.

2.2.1. *Caution.* To what does the Fourier series converge when x is not in the interval $(-\pi,\pi)$? When we build a Fourier series for a function defined on the interval $(-\pi,\pi)$, it is of the form:

$$\sum_{n\in\mathbb{Z}} c_n e^{inx}.$$

Each of the terms e^{inx} is a 2π periodic function. Hence the Fourier series is also a 2π periodic function. So, for $x = 2\pi$, the series does *not* converge to $e^{2\pi}$. Rather, it converges to e^0 because, writing

$$S(x) = \sum_{n\in\mathbb{Z}} c_n e^{inx}, \quad S(x+2k\pi) = S(x) \quad \forall k \in \mathbb{Z}.$$

For $x \in (-\pi,\pi)$, by the Theorem we proved, we have that $S(x) = e^x$. However, for x outside this interval, the series converges to the function which is equal to e^x on $(-\pi,\pi)$ and is extended to be 2π periodic.

2.3. **Example of the vibrating string.** Assume that at $t = 0$, the ends of the string are fixed, and we have pulled up the middle of it. This makes a shape which mathematically is described by the function

$$v(x) = \begin{cases} x, & 0 \le x \le \pi \\ 2\pi - x, & \pi \le x \le 2\pi \end{cases}$$

Assume that at $t = 0$ the string is not yet vibrating, so the initial conditions are then

$$\begin{cases} u(x,0) = v(x) \\ u_t(x,0) = 0 \end{cases}$$

We assume the ends of the string are fixed, so we have the boundary conditions

$$u(0) = u(2\pi) = 0.$$

The string is identified with the interval $[0,2\pi]$. Determine the function $u(x,t)$ which gives the height at the point x on the string at the time $t \ge 0$ which satisfies all these conditions.

We use our first technique, separation of variables. The wave equation demands that

$$\Box u = 0, \quad \Box u = \partial_{tt} u - \partial_{xx} u.$$

Write

$$u(x,t) = X(x)T(t).$$

Hit it with the wave equation:

$$X(x)T''(t) - X''(x)T(t) = 0.$$

We again *separate the variables* by dividing the whole equation by $X(x)T(t)$. Then we have

$$\frac{T''(t)}{T(t)} - \frac{X''(x)}{X(x)} = 0 \implies \frac{T''}{T} = \frac{X''}{X} = \text{ constant.}$$

The two sides depend on different variables, which makes them both have to be constant. We give that a name, λ. Then, since we have those handy dandy boundary conditions for X (but a much more complicated initial condition for $u(x,0) = v(x)$) we start with X. We solve

$$X'' = \lambda X, \quad X(0) = X(2\pi) = 0.$$

Exercise 2. *Show that the cases $\lambda \geq 0$ won't satisfy the boundary condition.*

We are left with $\lambda < 0$ which by our multivariable calculus theorem tells us that

$$X(x) = a\cos(\sqrt{|\lambda|}x) + b\sin(\sqrt{|\lambda|}x).$$

To get $X(0) = 0$, we must have $a = 0$. To get $X(2\pi) = 0$ we will need

$$\sqrt{|\lambda|}2\pi = k\pi \quad k \in \mathbb{Z}.$$

Hence

$$\sqrt{|\lambda|} = \frac{k}{2}, \quad k \in \mathbb{Z}.$$

Since $\sin(-x) = -\sin(x)$ are linearly dependent, we only need to take $k \in \mathbb{N}$ (without 0, you know, American $\mathbb{N}$). So, we have X which we index by n, writing

$$X_n(x) = \sin(nx/2) \quad n \in \mathbb{N}.$$

For now, we don't worry about the constant factor. Next, we have the equation for the partner-function (can't forget the partner function!)

$$\frac{T_n''}{T_n} = \lambda_n.$$

Since we know that $\lambda_n < 0$ and $\sqrt{|\lambda_n|} = n/2$ we have

$$\lambda_n = -\frac{n^2}{4}.$$

Hence, our handy dandy multivariable calculus theorem tells us that the solution

$$T_n(t) = a_n\cos(nt/2) + b_n\sin(nt/2).$$

Now, we have

$$u_n(x,t) = X_n(x)T_n(t), \quad \Box u_n = 0 \quad \forall n \in \mathbb{N}.$$

Hence, we also have

$$\Box \sum_{n\geq 1} u_n(x,t) = \sum_{n\geq 1} \Box u_n(x,t) = 0,$$

because $\Box$ is a linear partial differential operator. We don't know which of these u_n guys we need to build our solution according to the initial conditions, so we just take all of them for now and chuck them out later if we don't need them.

So, we now need
$$u(x,t) := \sum_{n\geq 1} u_n(x,t)$$
to satisfy the initial condition. The easiest of these is the one that has zero on the right, namely $u_t(x,0) = 0$. So, we differentiate $u(x,t)$ with respect to t and set $t = 0$,
$$u_t(x,t) = \sum_{n\geq 1} X_n(x) T_n'(0) = \sum_{n\geq 1} X_n(x) \left(-a_n \frac{n}{2} \sin(0) + b_n \frac{n}{2} \cos(0) \right)$$
$$= \sum_{n\geq 1} X_n(x) \frac{n}{2} b_n.$$
We need this to be the zero function. Basically, we are expanding the zero function in terms of the basis functions X_n. In the theorem for pointwise convergence of Fourier series, we used the c_n's. This was for convenience. The same theorem holds when we use the sine and cosine expansion like here. They're all equivalent. Of course, if we think about how to get the expansion in terms of the basis, the coefficients will be the scalar product of the zero function and X_n. This will be 0. So, the coefficients $b_n = 0$ for all n.

Now we use the other initial condition to get the coefficients a_n,
$$u(x,0) = \sum_{n\geq 1} X_n(x) a_n.$$
We want this to be equal to
$$v(x).$$
Although we've been working so far with the interval $[-\pi, \pi]$, this is basically the same. We are now on the interval $[0, 2\pi]$. The coefficients will be
$$\frac{1}{||X_n||^2} \langle v, X_n \rangle = \frac{1}{\pi} \int_0^{2\pi} X_n(x) v(x) dx.$$
What happened to the complex conjugation? Well, it is there, it just ain't doing nothin to v because v is real valued. Also, I leave it as an exercise to compute that
$$||X_n||^2 = \int_0^{2\pi} \sin(nx/2)^2 dx = \pi.$$
So, now is just to compute
$$\int_0^{\pi} \sin(nx/2) x dx + \int_{\pi}^{2\pi} \sin(nx/2)(2\pi - x) dx.$$
I leave this also as an exercise (you can use BETA :-)

2.4. Exercises for the week from [folland 1].

2.4.1. *Exercises to be demonstrated in the large group.*

(1) Compute the Fourier series of the function defined on $(-\pi, \pi)$
$$f(x) := \begin{cases} 0 & -\pi < x < 0 \\ x & 0 < x < \pi \end{cases}.$$
(2) Compute the Fourier series of the function defined on $(-\pi, \pi)$
$$f(x) := |\sin(x)|.$$

(3) Compute the Fourier series of the function defined on $(-\pi, \pi)$

$$f(x) := \begin{cases} 1 & -a < x < a \\ -1 & 2a < x < 4a \\ 0 & \text{elsewhere in } (-\pi, \pi). \end{cases}.$$

Here one ought to assume that $0 < a < \pi$ for this to make sense.

(4) Compute the Fourier series of the function defined on $(-\pi, \pi)$

$$f(x) = x^2.$$

2.4.2. *Exercises to be done by oneself.*

(1) Compute the Fourier series of the function defined on $(-\pi, \pi)$

$$f(x) := x(\pi - |x|).$$

(2) Compute the Fourier series of the function defined on $(-\pi, \pi)$

$$f(x) = e^{bx}.$$

(3) Use the Fourier series for the function $f(x) = |\sin(x)|$ to compute the sum

$$\sum_{n=1}^{\infty} \frac{1}{4n^2 - 1} = \frac{1}{2}, \quad \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{4n^2 - 1} = \frac{\pi - 2}{4}.$$

(4) Use the Fourier series for the function $f(x) = x(\pi - |x|)$ to compute the sum

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)^3} = \frac{\pi^3}{32}.$$

(5) Let $f(x)$ be the periodic function such that $f(x) = e^x$ for $x \in (-\pi, \pi)$, and extended to be 2π periodic on the rest of $\mathbb{R}$. Let

$$\sum_{n \in \mathbb{Z}} c_n e^{inx}$$

be its Fourier series. Therefore, by Theorem 2.1

$$e^x = \sum_{n \in \mathbb{Z}} c_n e^{inx}, \quad \forall x \in (-\pi, \pi).$$

If we differentiate this series term-wise then we get $\sum inc_n e^{inx}$. On the other hand, we know that $(e^x)' = e^x$. So, then we should have

$$\sum inc_n e^{inx} = \sum c_n e^{inx} \implies c_n = inc_n \quad \forall n.$$

This is clearly wrong. Where is the mistake?

2.4.3. *Exercises to be demonstrated in the small groups.*

(1) Use the Fourier series of the function $f(x) = x(\pi - |x|)$, defined on $(-\pi, \pi)$ and extended to be 2π periodic on $\mathbb{R}$, to compute the sums:

$$\sum_{n \geq 1} \frac{1}{n^2} = \frac{\pi^2}{6}, \quad \sum_{n \geq 1} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}.$$

(2) Use the Fourier series of the function $f(x) = e^{bx}$, defined on $(-\pi, \pi)$ and extended to be 2π periodic on $\mathbb{R}$, to compute the sum:

$$\sum_{n \geq 1} \frac{1}{n^2 + b^2} = \frac{\pi}{2b} \coth(b\pi) - \frac{1}{2b^2}.$$

(3) Use the Fourier series of the function $f(x) = x^2$, defined on $(-\pi, \pi)$ and extended to be 2π periodic on $\mathbb{R}$, to compute the sums:

$$x^2 - \pi^2 x = 12 \sum_{n \geq 1} \frac{(-1)^n \sin(nx)}{n^3}, \quad x \in (-\pi, \pi)$$

$$x^4 - 2\pi^2 x^2 = 48 \sum_{n \geq 1} \frac{(-1)^{n+1} \cos(nx)}{n^4} - \frac{7\pi^4}{15}$$

$$\sum_{n \geq 1} \frac{1}{n^4} = \frac{\pi^4}{90}.$$

2.4.4. *Exercises to be done by oneself.*

(1) Determine the Fourier sine and cosine series of the function

$$f(x) = \begin{cases} x & 0 \leq x \leq \frac{\pi}{2} \\ \pi - x & \frac{\pi}{2} \leq x \leq \pi \end{cases}$$

(2) Expand the function

$$f(x) = \begin{cases} 1 & 0 < x < 2 \\ -1 & 2 < x < 4 \end{cases}$$

in a cosine series on $[0, 4]$.

(3) Expand the function e^x in a series of the form

$$\sum_{n \in \mathbb{Z}} c_n e^{2\pi i n x}, \quad x \in (0, 1).$$

(4) Define

$$f(t) = \begin{cases} t & 0 \leq t \leq 1 \\ 1 & 1 < t < 2 \\ 3 - t & 2 \leq t \leq 3 \end{cases}$$

and extend f to be 3-periodic on $\mathbb{R}$. Expand f in a Fourier series. Determine, in the form of a Fourier series, a 3-periodic solution to the equation

$$y''(t) + 3y(t) = f(t).$$

References

[1] Gerald B. Folland, *Fourier Analysis and Its Applications,* Pure and Applied Undergraduate Texts Volume 4, (1992).