FOURIER ANALYSIS & METHODS

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Abstract. Caveat Emptor! These are just informal lecture notes. Errors are inevitable! Read at your own risk! Also, this is by no means a substitute for the textbook, which is warmly recommended: *Fourier Analysis and Its Applications,* by Gerald B. Folland. He was the first math teacher I had at university, and he is awesome. A brilliant writer. So, why am I even doing this? Good question...

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As a corollary to the theorem on the pointwise convergence of Fourier series we have

Corollary 1. *If f and g are 2π periodic and piecewise $\mathcal{C}^1$. Assume that at any point at which f is discontinuous, it satisfies*

$$f(x) = \frac{f(x_+) + f(x_-)}{2},$$

and the same is true for g. Then if f and g have the same Fourier coefficients, then $f = g$.

Proof: By assumption, f and g have the same Fourier series. Let us write the partial series

$$S_N(x) = \sum_{-N}^{N} c_n e^{inx}.$$

By the theorem on the pointwise convergence of Fourier series,

eq1day5

$$\lim_{N \to \infty} S_N(x) = \frac{f(x_+) + f(x_-)}{2} = \frac{g(x_+) + g(x_-)}{2}, \quad \forall x \in \mathbb{R}. \tag{1.1}$$

Now, at a point where f is continuous,

$$\frac{f(x_+) + f(x_-)}{2} = f(x).$$

Similarly, at a point where g is continuous

$$\frac{g(x_+) + g(x_-)}{2} = g(x).$$

So, by the assumptions on f and g, we have for all $x \in \mathbb{R}$

$$f(x) = \frac{f(x_+) + f(x_-)}{2}, \quad g(x) = \frac{g(x_+) + g(x_-)}{2}.$$

Thus, by (eq1day5 1.1),

$$f(x) = g(x) \quad \forall x \in \mathbb{R}.$$

1.1. **Differentiating and Integrating Fourier series.** First, let us demonstrate a fact about the Fourier series of a function and its derivative. Note that this is a theory item, so you may be asked to prove this on the exam.

Theorem 2. *Assume that f is 2π periodic, continuous, and piecewise $\mathcal{C}^1$. Let a_n, b_n, and c_n be the Fourier coefficients as we have defined them previously, and let a_n', b_n', c_n' be the Fourier coefficients of f' according to the same definition. Then we have*

$$a_n' = nb_n, \quad b_n' = -na_n, \quad c_n' = inc_n.$$

Proof: DO NOT DIFFERENTIATE THE FOURIER SERIES TERMWISE. To do this, you would need to prove that the series can be differentiated termwise, which at this point we do not have the techniques to demonstrate. So, it will be an incomplete and incorrect proof. Not a good thing.

Instead, use the definition of Fourier coefficients and integration by parts:

$$c_n' = \frac{1}{2\pi}\int_{-\pi}^{\pi} f'(x)e^{-inx}dx = \frac{1}{2\pi} f(x)e^{-inx}\Big|_{x=-\pi}^{x=\pi} - \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)(-ine^{-inx})dx$$

$$= \frac{in}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx = inc_n.$$

Above, we have used the fact that f is 2π periodic, and e^{-inx} is also 2π periodic so

$$\frac{1}{2\pi} f(x)e^{-inx}\Big|_{x=-\pi}^{x=\pi} = 0.$$

In the last step we use the definition of c_n. Recall that

$$a_n = c_n + c_{-n}, \quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx, \quad \forall n \in \mathbb{N}_{\geq 1},$$

and

$$b_n = i(c_n - c_{-n}), \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx, \quad \forall n \in \mathbb{N}_{\geq 1},$$

with

$$a_0 = c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx,$$

and the same relationship holds true for a_n', b_n', c_n'. We therefore compute

$$a_n' = c_n' + c_{-n}' = inc_n - inc_{-n} = in(c_n - c_{-n}) = nb_n,$$
$$b_n' = i(c_n' - c_{-n}') = i(inc_n + inc_{-n}) = -n(c_n + c_{-n}) = -na_n.$$

Now, using the theorem we have just proven, we obtain

Corollary 3. *Assume that f is 2π periodic, continuous, piecewise $\mathcal{C}^1$, and assume that f' is also piecewise $\mathcal{C}^1$. Then, if*

$$\sum_{-\infty}^{\infty} c_n e^{inx}$$

is the Fourier series for f, we have that

$$\sum_{n\in\mathbb{Z}} inc_n e^{inx}$$

is the Fourier series for f'.

Next, we shall demonstrate a small collection of results concerning the integration of Fourier series.

Exercise 1. *Show that if you compute the indefinite integrate*

$$\int e^{inx} dx, \quad n \in \mathbb{Z} \setminus \{0\},$$

the result is also a 2π *periodic function. What happens in the case* $n = 0$*?*

Before demonstrating the results concerning integration of Fourier series, it shall be useful to introduce a certain Hilbert space known as "little ell two."

Definition 4. *Let*

$$\ell^2(\mathbb{C}) := \{(z_n)_{n\in\mathbb{Z}}, \quad z_n \in \mathbb{C}\forall n, \text{ and } \sum_{n\in Z} |z_n|^2 < \infty\}.$$

This is a Hilbert space with the scalar product

$$\langle z, w\rangle := \sum_{n\in\mathbb{Z}} z_n \overline{w_n}, \quad z = (z_n)_{n\in Z}, \quad w = (w_n)_{n\in\mathbb{Z}}.$$

The norm on the Hilbert space, $\ell^2 = \ell^2(\mathbb{C})$ is defined by

$$||z|| = \sqrt{\sum_{n\in\mathbb{Z}} |z_n|^2}.$$

We also have a Cauchy-Schwarz inequality:

$$|\langle z, w\rangle| \le ||z||||w||.$$

We will use this together with the relationship between the Fourier coefficients for a piecewise $\mathcal{C}^1$ and continuous function, f, to prove

Theorem 5. *Assume that* f *is* 2π *periodic, continuous, and piecewise* $\mathcal{C}^1$*. Then the Fourier series of* f *converges absolutely uniformly to* f *on all of* $\mathbb{R}$*!*

Proof: By assumption, f' is piecewise continuous. Bessel's inequality tells us that

$$\sum_{\mathbb{Z}} |c_n'|^2 < \infty.$$

We use the preceding theorem to say that for all $n \neq 0$,

$$|c_n| = \left| c_n' \frac{1}{n} \right|.$$

Hence we can estimate

$$\sum_{n\in\mathbb{Z}} |c_n e^{inx}| = \sum_{n\in\mathbb{Z}} |c_n| = |c_0| + \sum_{n\in\mathbb{Z}\setminus 0} \frac{|c_n'|}{|n|}.$$

By Bessel's inequality

$$\sum_{n\in\mathbb{Z}} |c_n'|^2 < \infty,$$

and we know very well that

$$\sum_{n\in\mathbb{Z}\setminus 0} |n|^{-2} < \infty.$$

So, using the Cauchy-Schwarz inequality on ℓ^2, we have

$$\sum_{n\in\mathbb{Z}} |c_n| = |c_0| + \sum_{n\in\mathbb{Z}\setminus 0} \frac{|c_n'|}{|n|} \leq |c_0| + \sqrt{\sum_{n\in\mathbb{Z}\setminus 0} |c_n'|^2}\sqrt{\sum_{n\in\mathbb{Z}\setminus 0} |n|^{-2}} < \infty.$$

Therefore the Fourier series converges absolutely, and uniformly for all $x \in \mathbb{R}$, because we see that the convergence estimates are independent of the point x. Since the function is continuous, the limit of the series is, by the Theorem on the pointwise convergence of Fourier series

$$\frac{f(x_+) + f(x_-)}{2} = f(x).$$

We can repeat this idea to show that the more differentiable a function is, the faster its Fourier series converges.

Theorem 6. *Let f be 2π periodic, and assume that f is $\mathcal{C}^{k-1}$, and $f^{(k-1)}$ is piecewise $\mathcal{C}^1$, and f is piecewise $\mathcal{C}^k$. Then the Fourier coefficients of f satisfy*

$$\sum |n^k a_n|^2 < \infty, \quad \sum |n^k b_n|^2 < \infty, \quad \sum |n^k c_n|^2 < \infty.$$

If $|c_n| \leq c|n|^{-k-\alpha}$ for some $c > 0$ and $\alpha > 1$, for all $n \neq 0$, then $f \in \mathcal{C}^k$.

Proof: We apply the theorem relating the Fourier coefficients of f to those of the derivatives of f. Do it k times. We get

$$c_n^{(k)} = (in)^k c_n.$$

Next, we apply Bessel's inequality to conclude that since f is piecewise $\mathcal{C}^k$, $f^{(k)}$ is bounded on the interval hence it is in L^2 on the interval, and so

$$\sum |c_n^{(k)}|^2 < \infty.$$

Since

$$|c_n^{(k)}| = |n|^k |c_n|$$

this shows that

$$\sum |n^k c_n|^2 < \infty.$$

We have similar estimates for a_n and b_n using the same theorem, specifically

$$|a_n^{(k)}| = |n^k a_n|, \quad |b_n^{(k)}| = |n^k b_n|.$$

Hence,

$$\sum |n^k a_n| < \infty, \quad \sum |n^k b_n| < \infty.$$

Now we demonstrate the result which says that if the Fourier coefficients are sufficiently rapidly decaying, then the function f is actually in $\mathcal{C}^k$. Let

$$g(x) := f^{(k-1)}(x).$$

Then g is continuous and by assumption it is piecewise $\mathcal{C}^1$. Therefore, by the theorem on the pointwise convergence of Fourier series, the Fourier series of g converges to $g(x)$ for all x in $\mathbb{R}$. Next, we use the assumption and the fact that the Fourier coefficients of g are

$$c_n^{(k-1)} = (in)^{k-1} c_n.$$

Therefore

$$\sum_{n\in\mathbb{Z}} |c_n^{(k-1)} e^{inx}| = \left|c_0^{(k-1)}\right| + \sum_{n\neq 0} |n^{k-1}||c_n| \le \left|c_0^{(k-1)}\right| + c\sum_{n\neq 0} |n|^{k-1-k-\alpha} < \infty.$$

Hence, the series converges absolutely and uniformly in $\mathbb{R}$. Moreover, differentiating the series termwise is legitimate, because the result

$$\sum_{n\in\mathbb{Z}} inc_n^{(k-1)} e^{inx}$$

also converges absolutely and uniformly in $\mathbb{R}$:

$$\sum_{n\in\mathbb{Z}} |inc_n^{(k-1)}| \le \sum_{n\neq 0} |n||c_n^{(k-1)}| \le c\sum_{n\neq 0} |n||n|^{k-1-k-\alpha} < \infty$$

because $\alpha > 1$. Since the series is equal to $g(x) = f^{(k-1)}(x)$ for all $x \in \mathbb{R}$, and the series is a differentiable function for all $x \in \mathbb{R}$, this shows that g is differentiable for all $x \in \mathbb{R}$. Moreover, g' is continuous on $\mathbb{R}$, because the series defines a continuous function.[1] This is the case because the series defining g' converges absolutely and uniformly for all of $\mathbb{R}$. Hence, $f^{(k-1)}$ is in $\mathcal{C}^1$ on all of $\mathbb{R}$, and therefore f is in $\mathcal{C}^k$ on all of $\mathbb{R}$.

We now prove a theorem about integrating Fourier series.

Theorem 7. *Let f be a 2π periodic function which is piecewise continuous. Define*

$$F(x) := \int_0^x f(t)dt.$$

If $c_0 = 0$, then

$$F(x) = C_0 + \sum_{n\neq 0} \frac{c_n}{in} e^{inx}, \quad C_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} F(x)dx.$$

Similarly,

$$F(x) = \frac{1}{2}A_0 + \sum_{n\geq 1} \frac{a_n}{n}\sin(nx) - \frac{b_n}{n}\cos(nx).$$

Proof: We first note that F is continuous and piecewise $\mathcal{C}^1$, because it is the integral of a piecewise continuous function. Moreover, assuming $c_0 = 0$, we see that

$$F(x+2\pi) - F(x) = \int_0^{x+2\pi} f(t)dt - \int_0^x f(t)dt = \int_x^{x+2\pi} f(t)dt = \int_{-\pi}^{\pi} f(t)dt = 2\pi c_0 = 0.$$

Above we have used the nifty lemma that allows us to slide around integrals of periodic functions. So, F satisfies the assumptions of the theorem on pointwise convergence of Fourier series. We therefore have pointwise convergence of the Fourier series of F. Moreover, applying the theorem relating the Fourier coefficients of $F' = f$ to those of F, we have

$$C_n = \frac{c_n}{in} \quad n \neq 0.$$

[1]This is true because the series should really be viewed as the limit of the partial series, and each partial series defines a smooth, thus also continuous, function. The uniform limit of continuous functions is itself a continuous function.

(That's because $c_n = C_n'$ and the theorem says $C_n' = inC_n$ which shows $c_n = inC_n$, which we can re-arrange as above). Of course, the formula for C_0 is just the usual formula for it, because we can't say anything more specific without knowing more information on f. The re-statement in terms of a and b follows from the relationship between these and the c_n.

Remark 1. *If $c_0 \neq 0$, then define a new function*

$$g(t) := f(t) - c_0.$$

Since f is 2π periodic, so is g. Then, apply the theorem above to g. Note that

$$G(x) = \int_0^x g(t)dt = F(x) - c_0 x.$$

Moreover, the Fourier coefficients of g,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}(f(x) - c_0)e^{-inx}dx = c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx, \quad \forall n \neq 0.$$

So, the series for $G(x)$ from the theorem is

$$\widetilde{C_0} + \sum_{n\neq 0}\frac{c_n}{in}e^{inx},$$

with

$$\widetilde{C_0} = \frac{1}{2\pi}\int_{-\pi}^{\pi}(F(x) - c_0 x)\,dx = C_0.$$

So, in fact, it is the same C_0, where we have used the oddness of the function x above. Then, we get something of a corollary which says that in general, the series in the theorem,

$$C_0 + \sum_{n\neq 0}\frac{c_n}{in}e^{inx}, \quad C_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi}F(x)dx$$

converges to $F(x) - c_0 x$.

1.2. **Fourier sine and cosine series.** Let's say we are just looking at $[0,\pi]$. There are two ways to extend a function defined over there to all of $[-\pi,\pi]$. One way is oddly, and the other way is evenly. If we want to extend oddly, we define

$$f(x) := -f(-x), \quad x \in (-\pi, 0).$$

Then, we have computed in an exercise that the a_n coefficients are all zero, and the b_n coefficients are

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)dx.$$

Here we used the fact that sine is also an oddball. On the other hand, if we want to extend evenly, we define

$$f(x) := f(-x), \quad x \in (-\pi, 0).$$

Then, we have computed in an exercise that the b_n are all zero, because our function is even. Here we have the coefficients

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx = \frac{2}{\pi}\int_0^{\pi} f(x)\cos(nx)dx.$$

Above we used the fact that cosine is even. In this way, we may define Fourier sine and cosine series for functions on $[0, \pi]$. The Fourier sine series is defined to be

$$\sum_{n \geq 1} b_n \sin(nx), \quad b_n = \frac{2}{\pi} \int_0^\pi f(x) \sin(nx) dx$$

whereas the Fourier cosine series is

$$\frac{a_0}{2} + \sum_{n \geq 1} a_n \cos(nx), \quad a_n = \frac{2}{\pi} \int_0^\pi f(x) \cos(nx) dx, \quad \forall n \in \mathbb{N}.$$

Theorem 8. *Let f be a function which is piecewise $\mathcal{C}^1$ on $[0, \pi]$. Then the Fourier sine and cosine series converge to $f(x)$ for all $x \in (0, \pi)$ at which f is continuous. For other points, they converge to*

$$\frac{1}{2} \left(f(x_-) + f(x_+) \right).$$

Proof: First, we extend the function either evenly or oddly. Next, we extend it to all of $\mathbb{R}$ to be 2π periodic. Like Riker, we just *make it so.* We're only proving a statement about points in $(0, \pi)$. So, what happens outside of this set of points, well it don't matter. We apply the theorem on pointwise convergence of Fourier series now.

1.3. **How to compute neat sums using Fourier series and Theorem PWF$\sum$.**

Let us use a Fourier series to compute:

$$\sum_{n \geq 1} \frac{(-1)^n}{n^2 + b^2}.$$

To do this, we shall compute the Fourier series of the function defined to be e^{bx} for $|x| < \pi$ and extended to be 2π periodic. I do this by hand, but you are welcome to use Beta if it is helpful. We compute the coefficients directly:

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{bx} e^{-inx} dx = \frac{1}{2\pi(b - in)} e^{(b-in)\pi} - \frac{1}{2\pi(b - in)} e^{(b-in)(-\pi)}.$$

To simplify things, let us note that

$$e^{\pm in\pi} = (-1)^n.$$

Thus

$$c_n = \frac{1}{2\pi(b - in)} (-1)^n e^{b\pi} - \frac{1}{2\pi(b - in)} (-1)^n e^{-b\pi} = \frac{(-1)^n}{2\pi(b - in)} \left(e^{b\pi} - e^{-b\pi} \right) = \frac{(-1)^n}{\pi(b - in)} \sinh(b\pi).$$

The Fourier series is therefore

$$\frac{1}{\pi} \sinh(b\pi) \sum_{n \in \mathbb{Z}} \frac{(-1)^n}{b - in} e^{inx}.$$

Given the presence of the $(-1)^n$, which we also want, it makes sense to try computing with $x = 0$. The series is at this point

$$\frac{1}{\pi} \sinh(b\pi) \sum_{n \in \mathbb{Z}} \frac{(-1)^n}{b - in}.$$

Let us re-arrange things a wee bit:

$$\frac{1}{\pi}\sinh(b\pi)\sum_{n\in\mathbb{Z}}\frac{(-1)^n}{b-in}=\frac{\sinh(b\pi)}{\pi b}+\frac{1}{\pi}\sinh(b\pi)\sum_{n\geq 1}\frac{(-1)^n}{b-in}+\frac{1}{\pi}\sinh(b\pi)\sum_{n\leq 1}\frac{(-1)^n}{b-in}.$$

Let us re-write

$$\frac{1}{\pi}\sinh(b\pi)\sum_{n\leq 1}\frac{(-1)^n}{b-in}=\frac{1}{\pi}\sinh(b\pi)\sum_{n\geq 1}\frac{(-1)^n}{b+in},$$

with the observation that

$$(-1)^n=(-1)^{-n}.$$

Consequently the series is:

$$\frac{\sinh(b\pi)}{\pi b}+\frac{1}{\pi}\sinh(b\pi)\sum_{n\geq 1}\left(\frac{(-1)^n}{b-in}+\frac{(-1)^n}{b+in}\right)$$

$$=\frac{\sinh(b\pi)}{\pi b}+\frac{1}{\pi}\sinh(b\pi)\sum_{n\geq 1}(-1)^n\frac{b+in+b-in}{(b-in)(b+in)}$$

$$=\frac{\sinh(b\pi)}{\pi b}+\frac{1}{\pi}\sinh(b\pi)\sum_{n\geq 1}(-1)^n\frac{2b}{b^2+n^2}.$$

On the other hand, we use the theorem PWF$\sum$ to say that at the point $x=0$ the Fourier series of this function converges to

$$\frac{f(0_+)+f(0_-)}{2}.$$

At the point 0, note that our function is defined to be e^{bx} for $|x|<\pi$ and certainly $|0|<\pi$, so in particular, the function is continuous and thus the left and right limits are both equal and equal to $f(0)$ which is 1. Thus the series converges to 1, and so

$$1=\frac{\sinh(b\pi)}{\pi b}+\frac{1}{\pi}\sinh(b\pi)\sum_{n\geq 1}(-1)^n\frac{2b}{b^2+n^2}.$$

Re-arranging, we get

$$1-\frac{\sinh(b\pi)}{\pi b}=\frac{2b\sinh(b\pi)}{\pi}\sum_{n\geq 1}\frac{(-1)^n}{b^2+n^2}\implies\frac{\pi}{2b\sinh(b\pi)}-\frac{1}{2b^2}=\sum_{n\geq 1}\frac{(-1)^n}{b^2+n^2}.$$

1.3.1. *Exercises to be done by oneself: Hints.*

(1) Compute the Fourier series of the function defined on $(-\pi,\pi)$

$$f(x):=x(\pi-|x|).$$

Hint: Use Beta.

(2) Compute the Fourier series of the function defined on $(-\pi,\pi)$

$$f(x)=e^{bx}.$$

Hint: Use Beta.

(3) Use the Fourier series for the function $f(x) = |\sin(x)|$ to compute the sum

$$\sum_{n=1}^{\infty} \frac{1}{4n^2-1} = \frac{1}{2}, \quad \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{4n^2-1} = \frac{\pi-2}{4}.$$

Hint: use Beta to show that the Fourier series of the function defined to be $|\sin(x)|$ for $|x| < \pi$ and extended to be 2π periodic is:

$$\frac{2}{\pi} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\cos(2nx)}{4n^2-1}.$$

Use the theorem on the pointwise convergence of Fourier series to compute the value for $x = 0$. Then use algebra to obtain the value for

$$\sum_{n \geq 1} \frac{1}{4n^2-1}.$$

Next, take $x = \frac{\pi}{2}$, and proceed similarly to compute the sum

$$\sum_{n \geq 1} \frac{(-1)^{n+1}}{4n^2-1}.$$

(4) Use the Fourier series for the function $f(x) = x(\pi - |x|)$ to compute the sum

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)^3} = \frac{\pi^3}{32}.$$

Hint: use Beta to show that the Fourier series of the function $x(\pi - |x|)$ defined on $|x| < \pi$ and extended to be 2π periodic is:

$$\frac{8}{\pi} \sum_{n \geq 1} \frac{\sin(2n-1)x}{(2n-1)^3}.$$

To compute the sum, set $x = \frac{\pi}{2}$ and use the theorem on the pointwise convergence of Fourier series.

(5) Let $f(x)$ be the periodic function such that $f(x) = e^x$ for $x \in (-\pi, \pi)$, and extended to be 2π periodic on the rest of $\mathbb{R}$. Let

$$\sum_{n \in \mathbb{Z}} c_n e^{inx}$$

be its Fourier series. Therefore, by Theorem 2.1

$$e^x = \sum_{n \in \mathbb{Z}} c_n e^{inx}, \quad \forall x \in (-\pi, \pi).$$

If we differentiate this series term-wise then we get $\sum inc_n e^{inx}$. On the other hand, we know that $(e^x)' = e^x$. So, then we should have

$$\sum inc_n e^{inx} = \sum c_n e^{inx} \implies c_n = inc_n \quad \forall n.$$

This is clearly wrong. Where is the mistake?

Hint: What are the hypotheses of the theorem on differentiation of Fourier series (Theorem 2 in today's notes)? Are they all satisfied in this case?