FOURIER ANALYSIS & METHODS

JULIE ROWLETT

Abstract. Caveat Emptor! These are just informal lecture notes. Errors are inevitable! Read at your own risk! Also, this is by no means a substitute for the textbook, which is warmly recommended: *Fourier Analysis and Its Applications,* by Gerald B. Folland. He was the first math teacher I had at university, and he is awesome. A brilliant writer. So, why am I even doing this? Good question...

1. 2019.02.1

The methods for solving PDEs using

(1) Separation of variables (a means to an end)
(2) Superposition position
(3) Fourier series to find the initial data

are effective for *compact* geometric settings. In the context of $x \in \mathbb{R}$ this means that the problem is actually occurring on either $x \in [a, b]$ for some finite bounded interval $[a, b]$, or for $x \in \mathbb{S}^1$, we can identify this with the interval $[0, 2\pi]$ with the ends glued together. So again, it is a finite bounded set. In general these methods are effective as long as the geometry can fit in your pocket. To determine whether or not the question applies, ask the question, "Does the geometry fit in my pocket?" If it does, then yes. If not, like $\mathbb{R}$, then no.

1.1. Fourier series for any bounded interval.

To illustrate the method outlined above, we shall solve a typical exam problem:

$$u_{tt} = u_{xx}, \quad t > 0, \quad x \in (-1, 1),$$

$$\begin{cases} u(0,x) & = 1 - |x| \\ u_t(0,x) & = 0 \\ u_x(t,-1) & = 0 \\ u_x(t,1) & = 0 \end{cases}$$

We use separation of variables, writing $u(x,t) = X(x)T(t)$. It is just a means to an end. We write the PDE:

$$T''X = X''T.$$

Divide everything by XT to get

$$\frac{T''}{T} = \frac{X''}{X}.$$

Since the two sides depend on different variables, they are both constant. Start with the X side because we have more simple information about it. The boundary conditions that

$$u_x(t,-1) = u_x(t,1) = 0 \implies X'(-1) = X'(1) = 0.$$

So, we have the equation

$$\frac{X''}{X} = \text{ constant, call it } \lambda.$$

Thus we are solving

$$X'' = \lambda X, \quad X'(-1) = X'(1) = 0.$$

Case 1: $\lambda = 0$: In this case, we have solved this equation before. One way to think about it is like the second derivative is like acceleration. If $X'' = 0$, it's like saying X has constant acceleration. Therefore X can only be a linear function. Now, we have the boundary condition which says that $X'(-1) = X'(1) = 0$. So the slope of the linear function must be zero, hence X must be a constant function in this case. So, the only solutions in this case are the constant functions.

Case 2: $\lambda > 0$: In this case, a general solution is of the form:

$$X(x) = Ae^{\sqrt{\lambda}x} + Be^{-\sqrt{\lambda}x}.$$

Let us assume that A and B are not both zero. The left boundary condition requires

$$A\sqrt{\lambda}e^{-\sqrt{\lambda}} - \sqrt{\lambda}Be^{\sqrt{\lambda}} = 0.$$

Since $\lambda > 0$ we can divide by $\sqrt{\lambda}$ to say that we must have

$$Ae^{-\sqrt{\lambda}} = Be^{\sqrt{\lambda}} \implies \frac{A}{B} = e^{2\sqrt{\lambda}}.$$

The right boundary condition requires

$$A\sqrt{\lambda}e^{\sqrt{\lambda}} - \sqrt{\lambda}Be^{-\sqrt{\lambda}} = 0.$$

Since $\lambda > 0$, we can divide by $\sqrt{\lambda}$, to make this:

$$Ae^{\sqrt{\lambda}} = Be^{-\sqrt{\lambda}} \implies e^{2\sqrt{\lambda}} = \frac{B}{A}.$$

Hence combining with the other boundary condition we get:

$$\frac{A}{B} = e^{2\sqrt{\lambda}} = \frac{B}{A} \implies A^2 = B^2 \implies A = \pm B \implies \frac{A}{B} = \pm 1.$$

Neither of these are possible because

$$e^{2\sqrt{\lambda}} > 1 \text{ since } 2\sqrt{\lambda} > 0.$$

So, we run amok under the assumption that A and B are not both zero. Hence, the only solution in this case requires $A = B = 0$. This is the waveless wave.

Case 3: $\lambda < 0$: In this case a general solution is of the form:

$$X(x) = a\cos(\sqrt{|\lambda|}x) + b\sin(\sqrt{|\lambda|}x).$$

To satisfy the left boundary condition we need

$$-a\sqrt{|\lambda|}\sin(-\sqrt{|\lambda|}) + b\sqrt{|\lambda|}\cos(-\sqrt{|\lambda|}) = 0 \iff a\sin(\sqrt{|\lambda|}) = -b\cos(\sqrt{|\lambda|}).$$

To satisfy the right boundary condition we need

$$-a\sqrt{|\lambda|}\sin(\sqrt{|\lambda|}) + b\sqrt{|\lambda|}\cos(\sqrt{|\lambda|}) = 0 \iff a\sin(\sqrt{|\lambda|}) = b\cos(\sqrt{|\lambda|}).$$

Hence we need

eq:bc (1.1) $$a\sin(\sqrt{|\lambda|}) = -b\cos(\sqrt{|\lambda|}) = b\cos(\sqrt{|\lambda|}).$$

We do not want both a and b to vanish. So, we need to have either

(1) the sine vanishes, so we need $\sin(\sqrt{|\lambda|}) = 0$ which then implies that
$$\sqrt{|\lambda|} = n\pi, \quad n \in \mathbb{Z}$$
(2) or the cosine vanishes so we need $\cos(\sqrt{|\lambda|}) = 0$ which then implies that
$$\sqrt{|\lambda|} = \left(n + \frac{1}{2}\right)\pi, \quad n \in \mathbb{N}.$$

Note that these two cases are *mutually exclusive.* In case (1), by (1.1) this means that $b = 0$. In case (2), by (1.1) this means that $a = 0$. So, we have two types of solutions, which up to constant factor look like:
$$X_m(x) = \begin{cases} \cos(m\pi x/2) & m \text{ is even} \\ \sin(m\pi x/2) & m \text{ is odd} \end{cases}$$
In both cases,
$$\lambda_m = -\frac{m^2\pi^2}{4}.$$
We can now solve for the partner function, $T_m(t)$. The equation is
$$\frac{T_m''}{T_m} = \frac{X_m''}{X_m} = \lambda_m = -\frac{m^2\pi^2}{4}.$$
Therefore, we are in case 3 for the T_m function as well, so we know that
$$T_m(t) = a_m \cos\left(\frac{m\pi t}{2}\right) + b_m \sin\left(\frac{m\pi t}{2}\right).$$
Then we have for
$$u_m(x,t) = X_m(x)T_m(t), \quad \Box u_m = 0 \quad \forall m.$$
(Recall that $\Box = \partial_{tt} - \partial_{xx}$, that is the wave operator). Hence, our functions solve a homogeneous PDE, so we can use the superposition principle to smash them all together to make a super solution:
$$u(x,t) = \sum_{m\in\mathbb{N}} u_m(x,t) = \sum_{n\in\mathbb{N}} X_m(x)\left(a_m \cos\left(\frac{m\pi t}{2}\right) + b_m \sin\left(\frac{m\pi t}{2}\right)\right).$$
How do we determine the coefficients? Using the initial data and a Fourier series for it!!!

The initial data is
$$\begin{cases} u(0,x) & = 1 - |x| \\ u_t(0,x) & = 0 \end{cases}$$
Let us plug $t = 0$ into our solution:
$$u(x,0) = \sum_{m\in\mathbb{N}} X_m(x) a_m.$$
We demand that this is the initial data, so we need
$$1 - |x| = \sum_{m\in\mathbb{N}} X_m(x) a_m.$$
It is a Fourier series on the right side!! We therefore just need to expand the function $1 - |x|$ in a Fourier series. If we think about the basis functions $\{X_m(x)\}_{m\geq 0}$ then
$$a_m = \frac{\langle 1 - |x|, X_m(x)\rangle}{||X_m||^2},$$

where

$$\langle 1-|x|, X_m(x)\rangle = \int_{-1}^{1} (1-|x|)\overline{X_m(x)}dx,$$

$$||X_m||^2 = \int_{-1}^{1} |X_m(x)|^2 dx.$$

On an exam, you are not actually required to compute these integrals!

Now, for the other coefficients (the b_n), we use the condition on the derivative:

$$u_t(x,0) = \sum_{m\in\mathbb{N}} m_n \frac{m\pi}{2} X_m(x) = 0.$$

We know how to Fourier expand the zero function: its coefficients are all just zero. Hence, it suffices to take

$$b_m = 0 \forall m.$$

1.2. **Fourier series on an arbitrary interval.** When we use these tools to solve a PDE on a finite interval, as above, the initial data is *not* a periodic function. Moreover, it was not defined on the interval $(-\pi,\pi)$. The technique still works! It is actually quite beautiful. When we determined the coefficients, we solved for the Fourier coefficients on the interval $(-1,1)$. Here we explain how to do that in general.

For a function f defined on an interval $[a-\ell, a+\ell]$ for some $a\in\mathbb{R}$, and some $\ell>0$, we begin by extending f to be 2ℓ periodic on $\mathbb{R}$. Next, we define

$$g(t) := f\left(\frac{t\ell}{\pi}+a\right) = f(x),$$

that is

$$\frac{t\ell}{\pi}+a = x, \quad t = \frac{(x-a)\pi}{\ell}.$$

Then, the function $g(t)$ is 2π periodic, because

$$g(t+2\pi) = f\left(\frac{(t+2\pi)\ell}{\pi}+a\right) = f\left(\frac{t\ell}{\pi}+a+2\ell\right) = f\left(\frac{t\ell}{\pi}+a\right).$$

Above, we used the fact that f is 2ℓ periodic. If g is a reasonable function, ($\mathcal{L}^2$ will suffice), we can expand it into a Fourier series:

$$\sum_{n\in\mathbb{Z}} c_n e^{int},$$

with coefficients

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(t)e^{-int}dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} f\left(\frac{t\ell}{\pi}+a\right)e^{-int}dt.$$

Substituting in the integral,

$$x = \frac{t\ell}{\pi}+a, \quad dx = \frac{\ell dt}{\pi}$$

the coefficients become:

$$c_n = \frac{1}{2\pi}\frac{\pi}{\ell}\int_{a-\ell}^{a+\ell} f(x)e^{-in(x-a)\pi/\ell}dx = \frac{1}{2\ell}\int_{a-\ell}^{a+\ell} f(x)e^{-in(x-a)\pi/\ell}dx.$$

Then, we get by substituting for t in terms of x the Fourier series for f,

$$\sum_{n\in\mathbb{Z}} c_n e^{in\left(\frac{(x-a)\pi}{\ell}\right)}.$$

The same relationship holds for the Fourier cosine and sine coefficients:

$$a_0 = 2c_0, \quad a_n = c_n + c_{-n}, \quad b_n = i(c_n - c_{-n}), \quad n \geq 1,$$

or equivalently

$$a_n = \frac{1}{\ell}\int_{a-\ell}^{a+\ell} f(x)\cos(n(x-a)\pi/\ell)dx, \quad b_n = \frac{1}{\ell}\int_{a-\ell}^{a+\ell} f(x)\sin(n(x-a)\pi/\ell)dx,$$

and the Fourier series has the form

$$\frac{a_0}{2} + \sum_{n\geq 1} a_n\cos(n(x-a)\pi/\ell) + b_n\sin(n(x-a)\pi/\ell).$$

To what does the Fourier series converge?

Theorem 1. *Assume that f is defined on an interval $[a-\ell, a+\ell]$ for some $a\in\mathbb{R}$, and some $\ell > 0$, such that f is piecewise $\mathcal{C}^1$ on this interval. Then the Fourier series for f, defined by*

$$\sum_{n\in\mathbb{Z}} c_n e^{in\left(\frac{(x-a)\pi}{\ell}\right)}, \quad c_n = \frac{1}{2\ell}\int_{a-\ell}^{a+\ell} f(x)e^{-in(x-a)\pi/\ell}dx,$$

or equivalently the series

$$\frac{a_0}{2} + \sum_{n\geq 1} a_n\cos(n(x-a)\pi/\ell) + b_n\sin(n(x-a)\pi/\ell)$$

converges to $f(x)$ for all $x \in (a-\ell, a+\ell)$ at which f is continuous. At a point $x \in (a-\ell, a+\ell)$ where f is not continuous, the series converges to

eq:avg
$$\frac{f(x_+) + f(x_-)}{2}. \tag{1.2}$$

On the rest of $\mathbb{R}$, the series converges to a 2ℓ periodic function which is defined to be equal to (1.2) on $(a-\ell, a+\ell)$.

Exercise 1. *Prove the theorem. As a hint: apply the Theorem PCF$\sum$ to the function g above.*

1.3. **Concluding remarks for Fourier series.** When we are solving a PDE on a bounded interval, we may use this method:

(1) Separation of variables (a means to an end)
(2) Superposition position
(3) Fourier series to find the initial data.

When we use this method, the result, like from the preceding example, we obtained:

$$u(x,t) = \sum_{m\in\mathbb{N}} X_m(x)a_m\cos\left(\frac{m\pi t}{4}\right),$$

where

$$a_m = \frac{\langle 1-|x|, X_m\rangle}{||X_m||^2}.$$

This function solves the problem on our interval, $(-1,1)$. If we look at it *outside* the interval, this function will be 2-periodic. However, if we are solving the PDE on the interval $(-1,1)$, what happens outside this interval is of no importance.

Another application of Fourier series, viewed as its own topic in its own right (not just as something that appears from trying to solve PDEs) is to use Theorem 2.1 (Theorem PCF$\sum$) to compute nifty sums like:

$$\sum_{n\geq 1} \frac{1}{n^2}.$$

To do this, you need to find a Fourier series

$$\sum_{n\in\mathbb{Z}} c_n e^{inx}$$

for a function, f which is defined on $(-\pi,\pi)$ and extended 2π periodically. By substituting a specific value of x you want to recover the desired sum, eg $\sum n^{-2}$. For example, using the Fourier series for the function $f(x)=x^2$ on $(-\pi,\pi)$ and extended to be 2π periodic, you can use Theorem PCF$\sum$ to compute $\sum n^{-2}$.

1.4. **Hilbert spaces.** A *Hilbert space* is a complete[1], normed vector space whose norm is defined by a scalar product. The definition of a vector space means that if u and v are elements in your Hilbert space, then for all complex numbers a and b,

$$au+bv \text{ is in your Hilbert space.}$$

So, taking $a=b=0$, there is always a 0 vector in your Hilbert space. The fact that it is normed means that every element of the Hilbert space has a *length,* which is equal to its norm. To define this, we describe the scalar product. For a Hilbert space H, the scalar product satisfies:

$$u,v\in H \implies \langle u,v\rangle \in \mathbb{C},$$
$$c\in\mathbb{C} \implies \langle cu,v\rangle = c\langle u,v\rangle,$$
$$u,v,w\in H \implies \langle u+w,v\rangle = \langle u,v\rangle + \langle w,v\rangle,$$
$$\langle u,v\rangle = \overline{\langle v,u\rangle},$$
$$\langle u,u\rangle \geq 0, \quad = 0 \iff u=0.$$

Therefore, we can define the norm of a vector as

$$||u|| := \sqrt{\langle u,u\rangle}.$$

The norm of a vector is also equal to its distance from the 0 element of the Hilbert space. Similarly,

$$||u-v|| = \sqrt{\langle u-v, u-v\rangle}$$

is the distance between the elements u and v in your Hilbert space. We say that a set of elements

$$\{u_\alpha\} \subset H$$

is an orthonormal basis (ONB) for H if for any $v\in H$ there exist complex numbers (c_α) such that

$$v = \sum c_\alpha u_\alpha, \quad \langle u_\alpha, u_\beta\rangle = \delta_{\alpha,\beta} = \begin{cases} 1 & \alpha=\beta \\ 0 & \alpha\neq\beta. \end{cases}$$

[1]Every Cauchy sequence converges. Do you remember what a Cauchy sequence is? If not, please look it up or ask!

This is the Kronecker δ. You may be wondering why we haven't written an index for α. Well, that's because à priori, they could be uncountable.

Theorem 2. *A Hilbert space is separable if and only if it has either a finite ONB or a countable ONB.*

There is a cute proof here:
`http://www.polishedproofs.com/relationship-between-a-countable-orthonormal-basis-and-a-cou`
We're only going to be working with Hilbert spaces which have either a finite ONB or a countable ONB. The dimension of a Hilbert space is the number of elements in an ONB. Any finite dimensional Hilbert space is in bijection with the standard one

$$\mathbb{C}^n, \quad u, v \in \mathbb{C}^n \implies \langle u, v \rangle = u \cdot \overline{v}.$$

Thus, writing

$$u = (u_1, \ldots, u_n), \qquad \text{with each component } u_k \in \mathbb{C},\ k = 1, \ldots, n$$

and similarly for v,

$$\langle u, v \rangle = \sum_{k=1}^{n} u_k \overline{v_k}.$$

The bijection between any finite (n) dimensional Hilbert space and $\mathbb{C}^n$ comes from taking an ONB of the Hilbert space and mapping the elements of the ONB to the standard basis vectors of $\mathbb{C}^n$. Here are some useful basic results for Hilbert spaces.

Proposition 3. *Let H be a Hilbert space. For any u and v in H,*

$$||u+v||^2 = ||u||^2 + 2\Re\langle u, v \rangle + ||v||^2.$$

Proof: Compute:

$$||u+v||^2 = \langle u+v, u+v \rangle = \langle u, u+v \rangle + \langle v, u+v \rangle$$

$$= \langle u, u \rangle + \langle u, v \rangle + \langle v, v \rangle + \langle v, u \rangle$$

$$= ||u||^2 + \langle u, v \rangle + ||v||^2 + \overline{\langle u, v \rangle}.$$

We all know that for a complex number z,

$$z + \overline{z} = 2\Re(z).$$

So,

$$\langle u, v \rangle + \overline{\langle u, v \rangle} = 2\Re\langle u, v \rangle.$$

1.4.1. *Exercises to be done by oneself: Hints.*

(1) Determine the Fourier sine and cosine series of the function

$$f(x) = \begin{cases} x & 0 \le x \le \frac{\pi}{2} \\ \pi - x & \frac{\pi}{2} \le x \le \pi \end{cases}$$

Hint: Use either Beta or Folland Table 1 item 11 with $a = \frac{\pi}{2}$.

(2) Expand the function

$$f(x) = \begin{cases} 1 & 0 < x < 2 \\ -1 & 2 < x < 4 \end{cases}$$

in a cosine series on $[0, 4]$. Hint: Use either Beta or recall that in this case a Fourier cosine series will have coefficients

$$a_n = \frac{2}{\ell} \int_0^\ell f(x) \cos(n\pi x/\ell) dx,$$

where here the interval is $[0, \ell] = [0, 4]$ so $\ell = 4$. The Fourier cosine series is then of the form

$$\frac{a_0}{2} + \sum_{n \geq 1} a_n \cos(n\pi x/4).$$

(3) Expand the function e^x in a series of the form

$$\sum_{n \in \mathbb{Z}} c_n e^{2\pi i n x}, \quad x \in (0, 1).$$

Hint: Use Beta or us the results in the first part of today's notes. Here the center of the interval is the point

$$a = \frac{1}{2},$$

and the interval has length one so writing it as

$$[a - \ell, a + \ell] \implies a = \frac{1}{2} = \ell.$$

(4) Define

$$f(t) = \begin{cases} t & 0 \leq t \leq 1 \\ 1 & 1 < t < 2 \\ 3 - t & 2 \leq t \leq 3 \end{cases}$$

and extend f to be 3-periodic on $\mathbb{R}$. Expand f in a Fourier series. Determine, in the form of a Fourier series, a 3-periodic solution to the equation

$$y''(t) + 3y(t) = f(t).$$

Hint: Compute the Fourier coefficients by the methods from today, observing that the interval is $[0, 3]$ thus has center equal to the point $\frac{3}{2}$ and length 3, so writing the interval as

$$[a - \ell, a + \ell] \implies a = \frac{3}{2}, \quad \ell = \frac{3}{2}.$$

For the differential equation part, write

$$y(t) = \sum_{n \in \mathbb{Z}} y_n e^{in(t-a)\pi/\ell}, \quad a\frac{3}{2}, \quad \ell = \frac{3}{2}.$$

Pop this into the differential equation above and set it equal to the Fourier series for f. This will give you an equation for the coefficients y_n in terms of the Fourier coefficients of f.