FOURIER ANALYSIS & METHODS

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Abstract. Caveat Emptor! These are just informal lecture notes. Errors are inevitable! Read at your own risk! Also, this is by no means a substitute for the textbook, which is warmly recommended: *Fourier Analysis and Its Applications,* by Gerald B. Folland. He was the first math teacher I had at university, and he is awesome. A brilliant writer. So, why am I even doing this? Good question...

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1.1. **Why bother with Hilbert spaces?** Hilbert space theory is important because we can use it to rigorously justify using Fourier series to solve PDEs. Here is the main idea:

(1) Start with a PDE where the x variable is in a finite (bounded) interval.
(2) Separate variables by writing u, (the unsub) as a product like $u(x,t) = X(x)T(t)$. Plug it into the PDE.
(3) Solve for X using the boundary conditions. This will probably give lots of Xs which can be indexed by $\mathbb{N}$.
(4) Each X_n has a partner T_n. Solve for these. Probably, you've got some unknown constants.
(5) Is the PDE homogeneous? If so, $X_1T_1 + X_2T_2 + \ldots$ also solves the PDE so you can smash them together into a big party series. If *not* then you may need to do something else (i.e. steady state solution). In the homogeneous case, you will then use the IC and the collection $\{X_n\}$ to find the coefficients in T_n and end up with a solution of the form
$$\sum_{n\in\mathbb{N}} X_n(x)T_n(t).$$
It's precisely in this last step where the Hilbert space theory is being used to say that you can use the X_n obtain the IC, because the Hilbert space theory tells us when certain functions are basis functions for $\mathcal{L}^2$!

1.2. **Cauchy-Schwarz Inequality, Triangle Inequality, and Pythagorean Theorem.**

Proposition 1. *For any Hilbert space, H, for any u and v in H,*
$$|\langle u,v\rangle| \le ||u||||v||.$$

Proof: Assume that at least one of the two is non-zero. Let's assume $v \neq 0$, because otherwise we can just swap their names. We begin by considering the length of the vector u plus v scaled by a factor of t. If $t \to 0$, the length tends to $||u||^2$. What happens for other values of t? We compute it:
$$||u+tv||^2 = ||u||^2 + 2t\Re\langle u,v\rangle + t^2||v||^2, \quad t \in \mathbb{R}.$$

This is a real valued function of t. It's a quadratic function of t in fact. The derivative is

$$2t||v||^2 + 2\Re\langle u, v\rangle.$$

It's an upwards shaped quadratic function, so its unique minimum is when

$$t = -\frac{\Re\langle u, v\rangle}{||v||^2}.$$

If we then check out what happens at this value of t,

$$||u+tv||^2 = ||u||^2 - 2\frac{\Re\langle u, v\rangle}{||v||^2}\Re\langle u, v\rangle + \Re\langle u, v\rangle^2 \frac{||v||^2}{||v||^4} = ||u||^2 - \frac{\Re\langle u, v\rangle^2}{||v||^2}.$$

We know that

$$0 \le ||u+tv||^2$$

so we get

$$0 \le ||u||^2 - \frac{\Re\langle u, v\rangle^2}{||v||^2} \implies 0 \le ||u||^2||v||^2 - \Re\langle u, v\rangle^2.$$

This gives us

$$\Re\langle u, v\rangle^2 \le ||u||^2||v||^2.$$

Well, this is annoying because of that silly $\Re$. I wonder how we could make it turn into $|\langle u, v\rangle|$? Also, we don't want to screw up the $||u||^2||v||^2$ part. Well, we know how the scalar product interacts with complex numbers, for $\lambda \in \mathbb{C}$,

$$\langle \lambda u, v\rangle = \lambda\langle u, v\rangle.$$

So, if for example

$$\langle u, v\rangle = re^{i\theta}, r = |\langle u, v\rangle| \text{ and } \theta \in \mathbb{R}.$$

We can modify u, without changing $||u||$,

$$||e^{-i\theta}u|| = ||u||.$$

Moreover

$$\langle e^{-i\theta}u, v\rangle = e^{-i\theta}\langle u, v\rangle = e^{-i\theta}re^{i\theta} = |\langle u, v\rangle|.$$

So, if we repeat everything above replacing u with $e^{-i\theta}u$ we get

$$\Re\langle e^{-i\theta}u, v\rangle^2 \le ||e^{-i\theta}u||^2||v||^2 = ||u||^2||v||^2,$$

and by the above calculation

$$\langle e^{-i\theta}u, v\rangle = |\langle u, v\rangle| \in \mathbb{R} \implies \Re\langle e^{-i\theta}u, v\rangle^2 = |\langle u, v\rangle|^2.$$

So, we have

$$|\langle u, v\rangle|^2 \le ||u||^2||v||^2.$$

Taking the square root of both sides completes the proof of the Cauchy-Schwarz inequality.

We also have a triangle inequality.

Proposition 2. *For any u and v in a Hilbert space H,*

$$||u+v|| \le ||u|| + ||v||.$$

Proof: We just use the previous two results:

$$||u+v||^2 = ||u||^2 + 2\Re\langle u, v\rangle + ||v||^2 \leq ||u||^2 + 2||u||||v|| + ||v||^2 = (||u|| + ||v||)^2$$

so rooting we get the triangle inequality.

We have the Pythagorean theorem.

Proposition 3. *If u and v are orthogonal, then*

$$||u+v||^2 = ||u||^2 + ||v||^2.$$

Moreover, if $\{u_n\}_{n=1}^N$ are orthogonal, then

$$||\sum_{n=1}^N u_n||^2 = \sum_{n=1}^N ||u_n||^2.$$

Proof: The first statement follows from

$$||u+v||^2 = ||u||^2 + 2\Re\langle u, v\rangle + ||v||^2 = ||u||^2 + ||v||^2,$$

if u and v are orthogonal, because in that case their scalar product is zero. Moreover, for any collection of orthogonal vectors $\{u_1, \dots, u_n\}$ we proceed by induction. Assume that

$$||u_1 + \ldots + u_{n-1}||^2 = \sum_{k=1}^{n-1} ||u_k||^2.$$

Then, if u_n is orthogonal to all of $u_1, \dots, u_{n-1}$ we also have

$$\langle u_n, u_1 + \ldots + u_{n-1}\rangle = \langle u_n, u_1\rangle + \ldots + \langle u_n, u_{n-1}\rangle = 0 + \ldots + 0.$$

Hence u_n is also orthogonal to the sum,

$$\sum_{k=1}^{n-1} u_k.$$

By the Pythagorean theorem,

$$||u_n + \sum_{k=1}^{n-1} u_k||^2 = ||u_n||^2 + ||\sum_{k=1}^{n-1} u_k||^2.$$

By the induction assumption

$$= ||u_n||^2 + \sum_{k=1}^{n-1} ||u_k||^2 = \sum_{k=1}^{n} ||u_k||^2.$$

1.3. **Continuity of the scalar product.**

Proposition 4. *Using only the assumptions that the scalar product satisfies:*

$$\langle u, v \rangle = \overline{\langle v, u \rangle}$$

$$\langle au, v \rangle = a \langle u, v \rangle$$

$$\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$$

$$\langle u, u \rangle \geq 0, \quad \langle u, u \rangle = 0 \iff u = 0,$$

then the scalar product is a continuous function from $H \times H \to \mathbb{C}$.

Proof: It suffices to estimate

$$|\langle u, v \rangle - \langle u', v' \rangle| .$$

I would like to somehow get

$$u - u' \text{ and } v - v'.$$

So, well, just throw them in the first and last

$$\langle u - u', v \rangle = \langle u, v \rangle - \langle u', v \rangle.$$

That shows that

$$\langle u - u', v \rangle + \langle u', v \rangle = \langle u, v \rangle.$$

So, we see that

$$\langle u, v \rangle - \langle u', v' \rangle = \langle u - u', v \rangle + \langle u', v \rangle - \langle u', v' \rangle$$

We can smash the last two terms together because $-1 \in \mathbb{R}$ so

$$-\langle u', v' \rangle = \langle u', -v' \rangle \implies \langle u', v \rangle - \langle u', v' \rangle = \langle u', v - v' \rangle.$$

Hence,

$$|\langle u, v \rangle - \langle u', v' \rangle| = |\langle u - u', v \rangle + \langle u', v - v' \rangle| .$$

By the triangle inequality

$$|\langle u - u', v \rangle + \langle u', v - v' \rangle| \leq |\langle u - u', v \rangle| + |\langle u', v - v' \rangle| .$$

By the Cauchy-Schwarz inequality

$$|\langle u - u', v \rangle| + |\langle u', v - v' \rangle| \leq ||u - u'|| ||v|| + ||u'|| ||v - v'||.$$

We therefore see that for any fixed pair $(u, v) \in H \times H$, given $\epsilon > 0$, we can define

$$\delta := \min \left\{ \frac{\varepsilon}{2(||v|| + 1)}, \frac{\varepsilon}{2(||u|| + 1)}, 1 \right\}.$$

Then we estimate

$$||u - u'|| < \delta \implies ||u'|| < ||u|| + \delta \leq ||u|| + 1,$$

$$||u - u'|| ||v|| \leq \frac{\varepsilon ||v||}{2(||v|| + 1)} < \frac{\varepsilon}{2}.$$

and

$$||u'|| ||v - v'|| \leq \frac{(||u|| + 1)\varepsilon}{2(||u|| + 1)} \leq \frac{\varepsilon}{2},$$

so we obtain

$$|\langle u, v \rangle - \langle u', v' \rangle| < \varepsilon.$$

Remark 1. *This fact is useful because it allows us to bring limits inside the scalar product. You will see that we do this many times!*

1.4. **Bessel's inequality and the three equivalent conditions to be an ONB.** We prove a very useful inequality.

Theorem 5 (Bessel's Inequality for general Hilbert spaces)**.** *Let $\{\phi_n\}_{n\in\mathbb{N}}$ be an orthonormal set in a Hilbert space H. Then if $f \in H$,*

$$\sum_{n\in\mathbb{N}} |\langle f, \phi_n\rangle|^2 \le ||f||^2.$$

Moreover, the element

$$\sum_{n\in\mathbb{N}} \langle f, \phi_n\rangle\phi_n \in H.$$

Proof: By the Pythagorean theorem, for each $N \in \mathbb{N}$,

$$\|\sum_{n=1}^{N} \hat{f}_n\phi_n\|^2 = \sum_{n=1}^{N} |\hat{f}_n|^2.$$

Above, we have used the convenient notation

$$\hat{f}_n = \langle f, \phi_n\rangle.$$

In words, this is called the Fourier coefficient of f with respect to the orthonormal set (ONS) $\{\phi_n\}$. We compute that the square of the distance between f and its partial Fourier series

$$0 \le \|f - \sum_{n=1}^{N} \hat{f}_n\phi_n\|^2 = \|f\|^2 - 2\Re\langle f, \sum_{1}^{N} \hat{f}_n\phi_n\rangle + \|\sum_{1}^{N} \hat{f}_n\phi_n\|^2.$$

Let's look at the middle bit:

$$\langle f, \sum_{1}^{N} \hat{f}_n\phi_n\rangle = \sum_{1}^{N} \overline{\hat{f}_n}\langle f, \phi_n\rangle = \sum_{1}^{N} \overline{\hat{f}_n}\hat{f}_n = \sum_{1}^{n} |\hat{f}_n|^2.$$

Hence,

$$0 \le ||f||^2 - 2\sum_{1}^{N} |\hat{f}_n|^2 + \sum_{n=1}^{N} |\hat{f}_n|^2 = ||f||^2 - \sum_{1}^{N} |\hat{f}_n|^2$$

so re-arranging

$$\sum_{1}^{N} |\hat{f}_n|^2 \le ||f||^2.$$

Letting $N \to \infty$ completes the proof of Bessel's Inequality.

For the last part of the theorem, we will show that the sequence

$$\{F_N\}_{N\ge 1}, \quad F_N := \sum_{n=1}^{N} \hat{f}_n\phi_n$$

is a Cauchy sequence in H. Since Hilbert spaces are complete, it follows that this Cauchy sequence converges to a limit $F \in H$. So, let $\varepsilon > 0$ be given. Then, by Bessel's inequality, since

$$\sum_{1}^{\infty} |\hat{f}_n|^2 < \infty,$$

there exists $N \in \mathbb{N}$ such that

$$\sum_{N}^{\infty} |\hat{f}_n|^2 < \varepsilon^2.$$

This is because the tail of any convergent series can be made as small as we like. So, now if we have $N_1 \geq N_2 \geq N$, we estimate

$$||F_{N_1} - F_{N_2}||^2 = || \sum_{N_2+1}^{N_1} \hat{f}_n \phi_n ||^2 = \sum_{N_2+1}^{N_1} |\hat{f}_n|^2$$

$$\leq \sum_{N_2+1}^{\infty} |\hat{f}_n|^2 \leq \sum_{N}^{\infty} |\hat{f}_n|^2 < \varepsilon^2.$$

Consequently we have that for all $N_1 \geq N_2 \geq N$,

$$||F_{N_1} - F_{N_2}|| < \varepsilon.$$

This is the definition of being a Cauchy sequence.

Now, it turns out that the version of Bessel's inequality for the Fourier coefficients will actually be an equality, because $\{e^{inx}\}_{n \in \mathbb{Z}}$ is a basis for $\mathcal{L}^2$ on $[-\pi, \pi]$. In general, Bessel's inequality on a Hilbert space becomes an equality if and only if the orthonormal set $\{\phi_n\}$ is a basis.

1.5. **Proof of the 3 equivalent conditions to be an ONB in a Hilbert space.** This seems to be a fun one for some reason. It is rather nicely straightforward. Perhaps what makes it so nice is the pleasant setting of a Hilbert space, or translated directly from German, a Hilbert room. Hilbert rooms are cozy.

Theorem 6. *Let $\{\phi_n\}_{n \in \mathbb{N}}$ be orthonormal in a Hilbert space, H. TFAE (the following are equivalent):*

$$(1) \quad f \in H \text{ and } \langle f, \phi_n \rangle = 0 \forall n \in \mathbb{N} \implies f = 0.$$

$$(2) \quad f \in H \implies f = \sum_{n \in \mathbb{N}} \langle f, \phi_n \rangle \phi_n.$$

$$(3) \quad ||f||^2 = \sum_{n \in \mathbb{N}} |\langle f, \phi_n \rangle|^2 .$$

The last of these is known as Parseval's equation.

Proof: We shall proceed in order prove (1) $\implies$ (2), then (2) $\implies$ (3), and finally (3) $\implies$ (1). Just stay calm and carry on. So we begin by assuming (1) holds, and then we shall show that (2) must hold as well. First, we note that by Bessel's inequality, the series

$$\sum_{n \in \mathbb{N}} |\langle f, \phi_n \rangle|^2 \leq ||f||^2 < \infty.$$

Hence, if we know anything about convergent series, then we sure better know that the tail of the series tends to zero. The tail of the series is

$$\sum_{n \geq N} |\langle f, \phi_n \rangle|^2 \to 0 \text{ as } N \to \infty.$$

Now, let us define some elements in our Hilbert space, which we shall show comprise a Cauchy sequence. Let

$$g_N := \sum_{n=1}^{N} \langle f, \phi_n \rangle \phi_n.$$

For $M \geq N$, we have, using the Pythagorean Theorem and the orthonormality of the $\{\phi_n\}$,

$$||g_M - g_N||^2 = || \sum_{n=N+1}^{M} \langle f, \phi_n \rangle \phi_n ||^2 = \sum_{n=N+1}^{M} |\langle f, \phi_n \rangle|^2 \leq \sum_{n=N+1}^{\infty} |\langle f, \phi_n \rangle|^2 \to 0 \text{ as } N \to \infty.$$

Hence, by definition of Cauchy sequence (which one really should know at this point!), $\{g_N\}_{N \geq 1}$ is a Cauchy sequence in our Hilbert space. By definition of Hilbert space, every Hilbert space is complete. Thus every Cauchy sequence converges to a unique limit. Let us now call the limit of our Cauchy sequence, which is by definition,

$$\lim_{N \to \infty} g_N = \lim_{N \to \infty} \sum_{n=1}^{N} \langle f, \phi_n \rangle \phi_n = \sum_{n \in \mathbb{N}} \langle f, \phi_n \rangle \phi_n = g.$$

We will now show that $f - g$ satisfies

$$\langle f - g, \phi_n \rangle = 0 \forall n \in \mathbb{N}.$$

Then, because we are assuming (1) holds, this implies that $f - g = 0$, ergo $f = g$. So, we compute this inner product,

$$\langle f - g, \phi_n \rangle = \langle f, \phi_n \rangle - \langle g, \phi_n \rangle.$$

We insert the definition of g as the series,

$$\langle g, \phi_n \rangle = \langle \sum_{m \geq 1} \langle f, \phi_m \rangle \phi_m, \phi_n \rangle = \sum_{m \geq 1} \langle f, \phi_m \rangle \langle \phi_m, \phi_n \rangle = \langle f, \phi_n \rangle.$$

Above, we have used in the second equality the linearity of the inner product and the continuity of the inner product. In the third equality, we have used that $\langle \phi_m, \phi_n \rangle$ is 0 if $m \neq n$, and is 1 if $m = n$. Hence, only the term with $m = n$ survives in the sum. Thus,

$$\langle f - g, \phi_n \rangle = \langle f, \phi_n \rangle - \langle g, \phi_n \rangle = \langle f, \phi_n \rangle - \langle f, \phi_n \rangle = 0, \quad \forall n \in \mathbb{N}.$$

By (1), this shows that $f - g = 0 \implies f = g$.

Next, we shall assume that (2) holds, and we shall use this to demonstrate (3). Well, note that

$$f = \lim_{N \to \infty} g_N \implies ||f - g_N||^2 \to 0, \quad \text{as } N \to \infty.$$

Then, by the triangle inequality,

$$||f||^2 = ||f - g_N + g_N||^2 \leq ||f - g_N||^2 + ||g_N||^2 = ||f - g_N||^2 + \sum_{n=1}^{N} |\langle f, \phi_n \rangle|^2 \leq |f - g_N||^2 + \sum_{n \in \mathbb{N}} |\langle f, \phi_n \rangle|^2.$$

On the other hand, by Bessel's Inequality,

$$\sum_{n \in \mathbb{N}} |\langle f, \phi_n \rangle|^2 \leq ||f||^2.$$

So, we have a little sandwich, en smörgås, if you will, with $||f||^2$ right in the middle of our sandwich,

$$\sum_{n\in\mathbb{N}} |\langle f, \phi_n\rangle|^2 \leq ||f||^2 \leq ||f - g_N||^2 + \sum_{n\in\mathbb{N}} |\langle f, \phi_n\rangle|^2.$$

Letting $N \to \infty$ on the right side, the term $||f - g_N|| \to 0$, and so we indeed have

$$\sum_{n\in\mathbb{N}} |\langle f, \phi_n\rangle|^2 \leq ||f||^2 \leq \sum_{n\in\mathbb{N}} |\langle f, \phi_n\rangle|^2.$$

This of course means that all three terms are equal, because the terms all the way on the left and right side are the same!

Finally, we assume (3) holds and use it to show that (1) must also hold. This is pleasantly straightforward. We assume that for some f in our Hilbert space, $\langle f, \phi_n\rangle = 0$ for all n. Using (3), we compute

$$||f||^2 = \sum_{n\in\mathbb{N}} |\langle f, \phi_n\rangle|^2 = \sum_{n\in N} 0 = 0.$$

The only element in a Hilbert space with norm equal to zero is the 0 element. Thus $f = 0$.

1.6. **The Hilbert space $\mathcal{L}^2(-\pi, \pi)$.** It was a theorem that with the scalar product,

$$\langle f, g\rangle := \int_{-\pi}^{\pi} f(x)\overline{g(x)}dx,$$

the set of functions defined on $(-\pi, \pi)$ such that

$$\int_{-\pi}^{\pi} |f(x)|^2 dx < \infty$$

is a Hilbert space.[1] We also proved that the functions

$$\phi_n(x) := \frac{e^{inx}}{\sqrt{2\pi}} \text{ are an ONS.}$$

It turns out that they are in fact a basis. Note that they are not the only basis, because the functions

$$\{\frac{1}{\sqrt{2\pi}}, \frac{\sin(nx)}{\sqrt{\pi}}, \frac{\cos(nx)}{\sqrt{\pi}}\}_{n\geq 1} \text{ are also an ONS,}$$

and in fact they are an ONB. We will also see a bit later that we can build an ONB using polynomials.

[1]This is the workable definition, not the real one. We are sweeping measurability and equivalence up to sets of measure zero under the rug here.

1.7. **How to use Parseval's equation to compute sums.** We wish to compute the sum:

$$\sum_{n\geq 1} \frac{1}{(2n-1)^6}.$$

Often one would be given a hint, for example here, to use the Fourier series for $f(x) = x(\pi - |x|)$ on $(-\pi, \pi)$, extended to be 2π periodic. The Fourier series of this function is

$$\frac{8}{\pi} \sum_{n\geq 1} \frac{1}{(2n-1)^3} \sin((2n-1)x).$$

Let's express this in terms of the functions $\phi_n(x)$ as above. So, we unravel the sine as

$$\sin((2n-1)x) = \frac{e^{i(2n-1)x} - e^{-i(2n-1)x}}{2i}.$$

Hence the Fourier series is

$$\frac{8}{\pi} \sum_{n\geq 1} \frac{e^{i(2n-1)x}}{(2n-1)^3(2i)} - \frac{e^{-i(2n-1)x}}{(2n-1)^3(2i)}.$$

We therefore see that the coefficients

$$c_m = 0 \quad \forall m \in 2\mathbb{Z},$$

because there are no terms with e^{imx} for m even. We also see that the coefficients

$$c_{(2n-1)} = \frac{8}{\pi(2i)(2n-1)^3}, \quad n \in \mathbb{N}, \quad c_{-(2n-1)} = -\frac{8}{\pi(2i)(2n-1)^3}, \quad n \in \mathbb{N}.$$

Hence

$$|c_{2n-1}| = \left|c_{-(2n-1)}\right| = \frac{4}{\pi(2n-1)^3}.$$

Parseval's equation in this case says that:

$$\int_{-\pi}^{\pi} |f(x)|^2 dx = 2 \sum_{n\in\mathbb{N}} \left(\frac{4}{\pi(2n-1)^3} \right)^2 (2\pi).$$

The reason for the 2π on the right is because the functions $e^{i(2n-1)x}$ have $\mathcal{L}^2$ norm squared equal to 2π. So, to compute the sum we simply re-arrange:

$$\frac{\pi}{64} \int_{-\pi}^{\pi} |f(x)|^2 dx = \sum_{n\geq 1} \frac{1}{(2n-1)^6}.$$

Finally, we compute the integral:

$$\int_{-\pi}^{\pi} |f(x)|^2 dx = \int_{-\pi}^{\pi} x^2(\pi^2 - 2\pi|x| + |x|^2) dx = 2 \int_0^{\pi} x^2\pi^2 - 2x^3\pi + x^4$$

$$= 2\frac{\pi^5}{3} - 4\frac{\pi^5}{4} + 2\frac{\pi^5}{5} = \frac{\pi^5}{15}.$$

Thus

$$\frac{\pi^6}{(64) * (15)} = \sum_{n\geq 1} \frac{1}{(2n-1)^6}.$$

1.8. **Exercises for the week.** Those exercises from [folland 1] which shall be demonstrated are:

(1) (3.3.10.a) Evaluate the following series by applying Parseval's equation to certain Fourier expansions:
$$\sum_{n\geq 1} \frac{1}{n^4}.$$
(2) (3.3.10.b) Evaluate the following series by applying Parseval's equation to certain Fourier expansions:
$$\sum_{n\geq 1} \frac{1}{(2n-1)^6}$$
(3) (3.4.7.a) What is the best approximation in $\mathcal{L}^2$ to the function
$$f(x) = x \quad x \in [0, \pi]$$
among all functions of the form $a_0 + a_1 \cos x + a_2 \cos(2x)$?
(4) (3.5.4) Find all λ so that there exists a solution $f(x)$ defined on $[0, \ell]$ to the equation
$$f'' + \lambda f = 0, \quad f'(0) = 0, \quad f(\ell) = 0.$$
(5) (3.5.5) Find all solutions f on $[0, \ell]$ and all corresponding λ to the equation
$$f'' + \lambda f = 0, \quad f'(0) = \alpha f(0), \quad f'(\ell) = \beta f(\ell).$$
(6) (EO 23) Find all solutions f on $[0, a]$ and corresponding λ to the equation:
$$f'' + \lambda f = 0, \quad f(0) = f'(0), \quad f(a) = -2f'(a).$$
(7) (EO 24) Find all solutions f on $[0, 1]$ and all corresponding λ to the equation:
$$-e^{-4x} \frac{d}{dx} \left(e^{4x} \frac{du}{dx} \right) = \lambda u, \quad u(0) = 0, \quad u'(1) = 0.$$

Those exercises from [folland 1] which one should solve are:

(1) (3.3.1) Show that if $\{f_n\}_{n\geq 1}$ are elements of a Hilbert space, H, and we have for some $f \in H$ that
$$\lim_{n\to\infty} f_n = f,$$
then for all $g \in H$ we have
$$\lim_{n\to\infty} \langle f_n, g\rangle = \langle f, g\rangle.$$
(2) (3.3.2) Show that for all f, g in a Hilbert space one has
$$||\,||f|| - ||g||\,|| \leq ||f - g||.$$
(3) (3.3.10.c) Use Parseval's equation to compute
$$\sum_{n\geq 1} \frac{n^2}{(n^2+1)^2}.$$
(4) (3.4.7.b) What is the best approximation in $\mathcal{L}^2(0, \pi)$ to the function $f(x) = x$ amongst all functions of the form $b_1 \sin x + b_2 \sin(2x)$?
(5) (3.4.7.c) What is the best approximation in $\mathcal{L}^2(0, \pi)$ to the function $f(x) = x$ amongst all functions of the form $a\cos(x) + b\sin(x)$?

(6) (3.5.7) Find all solutions f on $[0,1]$ and all corresponding λ to the equation:

$$f'' + \lambda f = 0, \quad f(0) = 0, \quad f'(1) = -f(1).$$

(7) (3.4.2) Find constants a, b, A, B, C such that $f_0(x) = 1$, $f_1(x) = ax + b$, and $f_2(x) = Ax^2 + Bx + C$ are an orthonormal set in $L^2_w(0,\infty)$ where $w(x) = e^{-x}$.

(8) (4.2.3) Let $f(x)$ be the initial temperature at the point x in a rod of length ℓ, mathematicized as the interval $[0,\ell]$. Assume that heat is supplied at a constant rate at the right end, in particular $u_x(\ell,t) = A$ for a constant value A, and that the left end is held at the constant temperature 0, so that $u(0,t) = 0$. Find a series expansion for the temperature $u(x,t)$ such that the initial temperature is given by $f(x)$.

References

[1] Gerald B. Folland, *Fourier Analysis and Its Applications,* Pure and Applied Undergraduate Texts Volume 4, (1992).