FOURIER ANALYSIS & METHODS

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Abstract. Caveat Emptor! These are just informal lecture notes. Errors are inevitable! Read at your own risk! Also, this is by no means a substitute for the textbook, which is warmly recommended: *Fourier Analysis and Its Applications,* by Gerald B. Folland. He was the first math teacher I had at university, and he is awesome. A brilliant writer. So, why am I even doing this? Good question...

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The Hilbert space theory is useful for solving PDEs. We will see this connection through Sturm-Liouville problems. We've just got a little bit of theory to complete before we get to the SLPs. Before we begin, we shall define another Hilbert space.

Definition 1. *Let*

$$\ell^2(\mathbb{C}) := \{(z_n)_{n\in\mathbb{Z}}, \quad z_n \in \mathbb{C}\forall n, \text{ and } \sum_{n\in Z} |z_n|^2 < \infty\}.$$

This is a Hilbert space with the scalar product

$$\langle z, w\rangle := \sum_{n\in\mathbb{Z}} z_n \overline{w_n}, \quad z = (z_n)_{n\in Z}, \quad w = (w_n)_{n\in\mathbb{Z}}.$$

1.1. The Best Approximation Theorem.

This is another fun and cozy Hilbert room theory item.

Theorem 2. *Let $\{\phi_n\}_{n\in\mathbb{N}}$ be an orthonormal set in a Hilbert space, H. If $f \in H$, then*

$$||f - \sum_{n\in\mathbb{N}} \langle f, \phi_n\rangle \phi_n|| \le ||f - \sum_{n\in\mathbb{N}} c_n \phi_n||, \quad \forall \{c_n\}_{n\in\mathbb{N}} \in \ell^2,$$

and equality holds $\iff$ $c_n = \langle f, \phi_n\rangle$ is true $\forall n \in \mathbb{N}$.

Proof: We make a few definitions: let

$$g := \sum \widehat{f_n}\phi_n, \quad \widehat{f_n} = \langle f, \phi_n\rangle,$$

and

$$\varphi := \sum c_n \phi_n.$$

Exercise 1. *Prove that $\varphi \in H$. Note that you are not required to do this as part of the proof on an exam.*

Then we compute

$$||f - \varphi||^2 = ||f - g + g - \varphi||^2 = ||f - g||^2 + ||g - \varphi||^2 + 2\Re\langle f - g, g - \varphi\rangle.$$

I claim that

$$\langle f - g, g - \varphi\rangle = 0.$$

Just write it out (stay calm and carry on):

$$\langle f, g\rangle - \langle f, \varphi\rangle - \langle g, g\rangle + \langle g, \varphi\rangle$$

$$= \sum \overline{\widehat{f_n}}\langle f, \phi_n\rangle - \sum \overline{c_n}\langle f, \phi_n\rangle - \sum \widehat{f_n}\langle \phi_n, \sum \widehat{f_m}\phi_m\rangle + \sum \widehat{f_n}\langle \phi_n, \sum c_m\phi_m\rangle$$

$$= \sum |\widehat{f_n}|^2 - \sum \overline{c_n}\widehat{f_n} - \sum |\widehat{f_n}|^2 + \sum \widehat{f_n}\overline{c_n} = 0,$$

where above we have used the fact that ϕ_n are an orthonormal set. Then, we have

$$||f - \varphi||^2 = ||f - g||^2 + ||g - \varphi||^2 \geq ||f - g||^2,$$

with equality iff

$$||g - \varphi||^2 = 0.$$

Let us now write out what this norm is, using the definitions of g and φ. By their definitions,

$$g - \varphi = \sum(\widehat{f_n} - c_n)\phi_n.$$

By the Pythagorean theorem, due to the fact that the ϕ_n are an orthonormal set, and hence multiplying them by the scalars, $\widehat{f_n} - c_n$, they remain orthogonal, we have

$$||g - \varphi||^2 = \sum \left|\widehat{f_n} - c_n\right|^2.$$

This is a sum of non-negative terms. Hence, the sum is only zero if all of the terms in the sum are zero. The terms in the sum are all zero iff

$$\left|\widehat{f_n} - c_n\right| = 0 \forall n \iff c_n = \widehat{f_n} \forall n \in \mathbb{N}.$$

Corollary 3. *Assume that* $\{\phi_n\}$ *is an OS in a Hilbert space* H *such that*

$$\left\{\frac{\phi_n}{||\phi_n||}\right\}$$

is an ONB. Then the best approximation to $f \in H$ *of the form*

$$\sum_{n=1}^{N} c_n\phi_n$$

is given by taking

$$c_n = \frac{\langle f, \phi_n\rangle}{||\phi_n||^2}.$$

Exercise 2. *Prove this corollary using the best approximation theorem.*

We shall use but do not need to prove the following theorem.

Theorem 4. *The functions*

$$\left\{\frac{e^{inx}}{\sqrt{2\pi}}\right\}_{n\in\mathbb{Z}}$$

are an orthonormal basis for the Hilbert space $\mathcal{L}^2(-\pi, \pi)$*. The functions*

$$\left\{\frac{1}{\sqrt{2\pi}}, \quad \frac{\sin(nx)}{\sqrt{\pi}}, \quad \frac{\cos(nx)}{\sqrt{\pi}}\right\}_{n\in\mathbb{N}}$$

are also an ONB for this Hilbert space.

The proof cannot be rigorously completed because we are lacking some fundamental results from more advanced analysis. However, we can at least explain how the proof goes, up to the details which require measure theory. It is actually rather simple. Let f be in $\mathcal{L}^2$. If f is continuous and piecewise $\mathcal{C}^1$ on $(-\pi, \pi)$ then the Theorem PCF$\sum$ says that

$$\lim_{N\to\infty} S_N(x) = f(x) \quad \forall x \in (-\pi, \pi),$$

where we note here that the Fourier series

$$S_N(x) = \sum_{-N}^{N} c_n e^{inx} = \sum_{-\mathbb{N}}^{N} \hat{f}_n \phi_n(x), \quad \phi_n(x) := \frac{e^{inx}}{\sqrt{2\pi}}, \quad \hat{f}_n := \int_{-\pi}^{\pi} f(x)\overline{\phi_n(x)}dx.$$

So, for all such f, we have the equality

$$\sum_{n\in\mathbb{Z}} \hat{f}_n \phi_n(x) = f(x), \quad \forall x \in (-\pi, \pi).$$

This is the second necessary and sufficient condition required for the functions $\{\phi_n\}$ to be a basis for the Hilbert space. For general f, this is where we use a fact that goes beyond the scope of the course. The fact says that continuous, piecewise $\mathcal{C}^1$ functions can be used to approximate all $\mathcal{L}^2$ functions. So, using this more advanced mathematical fact, the result for continuous piecewise $\mathcal{C}^1$ functions implies the same result for all $\mathcal{L}^2$ functions.

1.1.1. *Application of the best approximation theorem.* The type of problem one might be asked here is to find the numbers $\{c_j\}_{j=0}^3$ so that

$$||f - \sum_{j=0}^{3} c_j \phi_j||^2$$

is minimized, where ϕ_j is as above, and

$$f(x) = \begin{cases} 0 & -\pi < x < 0 \\ 1 & 0 \le x \le \pi \end{cases}$$

So, basically we compute the coefficients. One must only pay careful attention to whether or not the functions (here ϕ_j) are normalized. In general, one computes

$$c_j = \frac{\langle f, \phi_j \rangle}{||\phi_j||^2}.$$

So if $||\phi_j||^2 = 1$, then dividing by this does no harm. Here, we therefore compute

$$c_j = \int_{-\pi}^{\pi} f(x) \frac{e^{-ijx}}{\sqrt{2\pi}} dx = \begin{cases} \frac{\pi}{\sqrt{2\pi}} & j = 0 \\ -\frac{(-1)^j}{j\sqrt{2\pi}} + \frac{1}{j\sqrt{2\pi}} & j = 1, 2, 3 \end{cases}$$

1.2. **Spectral Theorem Motivation.** Basically, a linear (partial or ordinary) differential operator with constant coefficients will act on a certain Hilbert space. For example, the operator

$$\Delta = -\partial_x^2$$

acts on the Hilbert space H^2. Don't worry about what it is precisely, because what's important is just that it's a Hilbert space, and it happens to sit inside the

Hilbert space $\mathcal{L}^2$. This operator takes elements of the Hilbert space H^2 and sends them to the Hilbert space $\mathcal{L}^2$. It is a linear operator because

$$\partial_x^2(f(x) + g(x)) = f''(x) + g''(x) = \partial_x^2(f(x)) + \partial_x^2(g(x)).$$

So if we think of the functions as vectors, then Δ is like a linear map that takes in vectors and spits out vectors. Just like linear maps on finite dimensional vector spaces, which can be represented by a matrix, a linear operator on a Hilbert space can be represented by a matrix. If it is a sufficiently "nice" operator, then there will exist an orthonormal basis of eigenfunctions with corresponding eigenvalues. Here it is useful to recall

Theorem 5 (Spectral Theorem for $\mathbb{C}^n$)**.** *Assume that A is a Hermitian matrix. Then there exists an orthonormal basis of $\mathbb{C}^n$ which consists of eigenvectors of A. Moreover, each of the eigenvalues is real.*

Proof: Remember what Hermitian means. It means that for any $u, v \in \mathbb{C}^n$, we have

$$\langle Au, v\rangle = \langle u, Av\rangle.$$

By the Fundamental Theorem of Algebra, the characteristic polynomial

$$p(x) := \det(A - xI)$$

factors over $\mathbb{C}$. The roots of p are $\{\lambda_k\}_{k=1}^n$. These are by definition the eigenvalues of A. First, we consider the case when A has zero as an eigenvalue. If this is the case, then we define

$$K_0 := \mathrm{Ker}(A) = \{u \in \mathbb{C}^n : Au = 0\}.$$

We note that all nonzero $u \in K_0$ are eigenvectors of A for the eigenvalue 0. Since K_0 is a k-dimensional subspace of $\mathbb{C}^n$, it has an ONB $\{v_1, \ldots, v_k\}$. If $k = n$, we are done. So, assume that $k < n$. Then we consider

$$K_0^\perp = \{u \in \mathbb{C}^n : \langle u, v\rangle = 0 \forall v \in K_0\}.$$

Note that if $u \in K_0^\perp$ then

$$\langle Au, v\rangle = \langle u, Av\rangle = 0 \quad \forall v \in K_0.$$

Hence $A : K_0^\perp \to K_0^\perp$. Moreover, if

$$u \in K_0^\perp, \quad Au = 0 \implies u \in K_0 \cap K_0^\perp \implies u = 0.$$

Hence A is bijective from $K_0^\perp$ to itself. Since A has eigenvalues $\{\lambda_j\}_{j=1}^n$, and 0 appears with multiplicity k, $\lambda_{k+1} \neq 0$. It has some non-zero eigenvector. Let's call it u. Since it is an eigenvector it is not zero, so we define

$$v_{k+1} := \frac{u}{||u||}.$$

Proceeding inductively, we define K_1 to be the span of the vectors $\{v_1, \ldots, v_{k+1}\}$. We look at A restricted to $K_1^\perp$. We note that A maps K_1 to itself because if

$$v = \sum_1^{k+1} c_j v_j \implies Av = \sum_1^{k+1} c_j A v_j = \sum_1^{k+1} c_j \lambda_j v_j \in K_1.$$

Similarly, if $w \in K_1^\perp$,

$$\langle Aw, v\rangle = \langle w, Av\rangle = 0 \forall v \in K_1.$$

So, A maps $K_1^\perp$ into itself. Since the kernel of A is in K_1, A is a surjective and injective map from $K_1^\perp$ into itself. We note that A restricted to $K_1^\perp$ satisfies the same hypotheses as A, in the sense that it is still Hermitian, and it has a characteristic polynomial of degree equal to the dimension of $K_1^\perp$ So, there is an eigenvalue λ_{k+2}, for A as a linear map from $K_1^\perp$ to itself. It has an eigenvector, which we may assume has unit length, contained in $K_1^\perp$. Call it v_{k+2}. Continue inductively until we reach in this way $\{v_1, \ldots, v_n\}$ to span $\mathbb{C}^n$.

Why are the eigenvalues all real? This follows from the fact that if λ is an eigenvalue with eigenvector u then

$$\langle Au, u\rangle = \lambda ||u||^2 = \langle u, Au\rangle = \overline{\lambda}||u||^2.$$

Since u is an eigenvector it is not zero, so this forces $\lambda = \overline{\lambda}$.

1.3. **An example.** Let us do an example. On $[-\pi, \pi]$, the functions which satisfy

$$\Delta f = \lambda f, \quad f(-\pi) = f(\pi)$$

are

$$f(x) = f_n(x) = e^{inx}.$$

The corresponding

$$\lambda_n = n^2.$$

So, the eigenvalues of Δ with this particular boundary condition are n^2, and the corresponding eigenfunctions are $e^{\pm inx}$. We have proven that these are orthogonal. We can make them orthonormal by dividing by the norms,

$$\left\{ \frac{e^{inx}}{\sqrt{2\pi}} \right\}_{n\in\mathbb{Z}}.$$

We note that for all f and g in $\mathcal{L}^2$ which satisfy $f(-\pi) = f(\pi)$, $g(-\pi) = g(\pi)$ and which are also (at least weakly) twice differentiable, we would also get $f'(-\pi) = f'(\pi)$ and similarly for g, so that

$$\langle \Delta f, g\rangle = \int_{-\pi}^{\pi} -f''(x)\overline{g(x)}dx = -f'(x)\overline{g(x)}\Big|_{-\pi}^{\pi} + \int_{-\pi}^{\pi} f'(x)\overline{g'(x)}dx$$

$$= -f'(x)\overline{g(x)}\Big|_{-\pi}^{\pi} + f(x)\overline{g'(x)}\Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} f(x)\overline{g''(x)}dx.$$

Due to the boundary conditions, all that survives is

$$-\int_{-\pi}^{\pi} f(x)\overline{g''(x)}dx = \langle f, \Delta g\rangle.$$

So we see that

$$\langle \Delta f, g\rangle = \langle f, \Delta g\rangle.$$

This is just like the spectral theorem for Hermitian matrices! There is a similar spectral theorem here, a "grown-up linear algebra" theorem, called The Spectral Theorem. This grown-up version of the spectral theorem says that, like a Hermitian matrix, the operator Δ also has an $\mathcal{L}^2$ orthonormal basis of eigenfunctions. Hence, by this theorem, we know that the orthonormal *set,*

$$\left\{ \frac{e^{inx}}{\sqrt{2\pi}} \right\}_{n\in\mathbb{Z}},$$

(which à priori could be missing stuff) is in fact not missing anything, spans all of $\mathcal{L}^2$, and is an ONB. If you're interested in this topic, you can try to convince me to give a PhD/Master's course on it. With sufficiently many interested students, I may be convinced.

1.4. **Regular SLPs.** Let L be a linear, second order ordinary differential operator. So, we can write

$$L(f) = r(x)f''(x) + q(x)f'(x) + p(x)f(x).$$

Above, r, q, and p are specified REAL VALUED functions. As a simple example, take $r(x) = -1$, and $q(x) = p(x) = 0$. Then we have

$$L(f) = \Delta f = -f''(x).$$

We are working with functions defined on an interval $[a, b]$ which is a *finite* interval. So, the Hilbert space in which everything is happening is $\mathcal{L}^2$ on that interval. Like with matrices, we can think about the *adjoint* of the operator L. The adjoint by definition satisfies

$$\langle Lf, g\rangle = \langle f, L^* g\rangle,$$

where we are using L^* to denote the adjoint operator. Whatever it is. On the left side, we know what everything is, so we write it out by definition of the scalar product

$$\langle Lf, g\rangle = \int_a^b L(f)\overline{g(x)}dx = \int_a^b \left(r(x)f''(x) + q(x)f'(x) + p(x)f(x)\right)\overline{g(x)}dx.$$

Integrating by parts, we get

$$= (r\bar{g})f'|_a^b - \int_a^b (r\bar{g})'f' + (qg)f|_a^b - \int_a^b (q\bar{g})'f + \int_a^b pf\bar{g}$$

$$= (r\bar{g})f' + (q\bar{g})f|_a^b - \int_a^b \left[(r\bar{g})'f' + (q\bar{g})'f - pf\bar{g}\right].$$

We integrate by parts once more on the $(r\bar{g})'f'$ term to get

$$= (r\bar{g})f' - (r\bar{g})'f + (q\bar{g})f)|_a^b + \int_a^b (r\bar{g})''f - (q\bar{g})'f + fp\bar{g}.$$

So, if the boundary conditions are chosen to make the stuff evaluated from a to b (these are called the boundary terms in integration by parts) vanish, then we could define

$$L^* g = (rg)'' - (qg)' + pg,$$

since then

$$\langle Lf, g\rangle = \int_a^b (r\bar{g})''f - (q\bar{g})'f + fp\bar{g} = \langle f, L^* g\rangle.$$

Here we use that r, q and p are real valued functions, so $\bar{r} = r$, $\bar{q} = q$, and $\bar{p} = p$. For the spectral theorem to work, we will want to have

$$L = L^*.$$

When this holds, we say that L *is formally self-adjoint.* So, we need

$$Lf = L^* f \iff rf'' + qf' + pf = (rf)'' - (qf)' + pf.$$

We write the things out:

$$rf''+qf'+pf = (rf'+r'f)'-qf'-q'f+pf \iff rf''+qf' = rf''+2r'f'+r''f-qf'-q'f$$

$$\iff qf' = 2r'f' + r''f - qf' - q'f \iff (2q - 2r')f' + (r'' - q')f = 0.$$

To ensure this holds for all f, we set the coefficient functions equal to zero:

$$2q - 2r' = 0 \implies q = r', \quad q' = r''.$$

Well, that just means that $q = r'$. So, we need L to be of the form

$$Lf = rf'' + r'f' + pf = (rf')' + pf.$$

The boundary terms should also vanish, so we want:

$$(r\bar{g})f' - (r\bar{g})'f + (q\bar{g})f)\big|_a^b = (r\bar{g})f' - (r\bar{g})'f + (r'\bar{g})f\big|_a^b = 0,$$

$$\iff r\bar{g}f' - r'\bar{g}f - r\bar{g}'f + r'\bar{g}f\big|_a^b = 0 \iff r\bar{g}f' - r\bar{g}'f\big|_a^b = 0$$

$$\iff r(\bar{g}f' - \bar{g}'f)\big|_a^b = 0.$$

So, it suffices to assume that we are working with functions f and g that satisfy

$$(\bar{g}f' - \bar{g}'f)\big|_a^b = 0.$$

Writing this out we get:

$$\bar{g}(b)f'(b) - \bar{g}'(b)f(b) - (\bar{g}(a)f'(a) - \bar{g}'(a)f(a)) = 0 \iff$$

$$\bar{g}(b)f'(b) - \bar{g}'(b)f(b) = \bar{g}(a)f'(a) - \bar{g}'(a)f(a).$$

This is how we get to the definition of a regular SLP on an interval $[a, b]$. It is specified by

(1) a formally self-adjoint operator

$$L(f) = (rf')' + pf,$$

where r and p are real valued, r, r', and p are continuous, and $r > 0$ on $[a, b]$.

(2) self-adjoint boundary conditions:

$$B_i(f) = \alpha_i f(a) + \alpha_i' f'(a) + \beta_i f(b) + \beta_i' f'(b) = 0, \quad i = 1, 2.$$

The self adjoint condition further requires that the coefficients $\alpha_i, \alpha_i', \beta_i, \beta_i'$ are such that for all f and g which satisfy these conditions

$$r(\bar{g}f' - \bar{g}'f)\big|_a^b = 0.$$

(3) a positive, continuous function w on $[a, b]$.

The SLP is to find all solutions to the BVP

$$L(f) + \lambda wf = 0, \quad B_i(f) = 0, \quad i = 1, 2.$$

The eigenvalues are all numbers λ for which there exists a corresponding non-zero eigenfunction f so that together they satisfy the above equation, and f satisfies the boundary condition.

We then have a miraculous fact.

Theorem 6 (Adult Spectral Theorem)**.** *For every regular Sturm-Liouville problem as above, there is an orthonormal basis of L^2_w consisting of eigenfunctions $\{\phi_n\}_{n\in\mathbb{N}}$ with eigenvalues $\{\lambda_n\}_{n\in\mathbb{N}}$. We have*

$$\lim_{n\to\infty} \lambda_n = \infty.$$

Here, L^2_w is the weighted Hilbert space consisting of (the almost everywhere-equivalence classes of measurable) functions on the interval $[a, b]$ which satisfy

$$\int_a^b |f(x)|^2 w(x) dx < \infty,$$

and the scalar product is

$$\langle f, g\rangle_w = \int_a^b f(x)\overline{g(x)} w(x) dx.$$

We are not equipped to prove this fact. You can rest assured however that it is done through the techniques of functional analysis and bears similarity to the proof of the spectral theorem for finite dimensional vector spaces.

1.5. **Exercises for the week: Hints.** Those exercises from [1] which one should solve are:

(1) (3.3.1) Show that if $\{f_n\}_{n\geq 1}$ are elements of a Hilbert space, H, and we have for some $f \in H$ that

$$\lim_{n\to\infty} f_n = f,$$

then for all $g \in H$ we have

$$\lim_{n\to\infty} \langle f_n, g\rangle = \langle f, g\rangle.$$

Hint: Apply the Cauchy-Schwarz inequality to $\langle f_n - f, g\rangle$.

(2) (3.3.2) Show that for all f, g in a Hilbert space one has

$$|||f|| - ||g||| \leq ||f - g||.$$

Hint: First show that for any real numbers a and b,

$$|a - b|^2 = a^2 - 2ab + b^2.$$

Next, apply this fact with $a = ||f||$ and $b = ||g||$ to show that

$$|||f|| - ||g||| = ||f||^2 - 2||f||||g|| + ||g||^2.$$

Compare this to

$$||f - g||^2 = ||f||^2 - 2\Re\langle f, g\rangle + ||g||^2.$$

(3) (3.3.10.c) Use Parseval's equation to compute

$$\sum_{n\geq 1} \frac{n^2}{(n^2+1)^2}.$$

Hint: The Fourier series for the function $\sinh x$ on $(-\pi, \pi)$ and extended to be 2π periodic is

$$\frac{2\sinh\pi}{\pi} \sum_{n\geq 1} \frac{(-1)^{n+1} n}{n^2+1} \sin(nx).$$

(4) (3.4.7.b) What is the best approximation in $\mathcal{L}^2(0, \pi)$ to the function $f(x) = x$ amongst all functions of the form $b_1 \sin x + b_2 \sin(2x)$? Hint: By the best approximation theorem, unnormed version, compute

$$b_j = \frac{1}{||\sin jx||^2} \int_0^\pi x \sin(jx) dx, \quad j = 1, 2.$$

(5) (3.4.7.c) What is the best approximation in $\mathcal{L}^2(0,\pi)$ to the function $f(x) = x$ amongst all functions of the form $a\cos(x) + b\sin(x)$? Hint: By the best approximation theorem, unnormed version, compute

$$a = \frac{1}{||\cos(x)||^2}\int_0^\pi x\cos(x)dx, \quad b = \frac{1}{||\sin(x)||^2}\int_0^\pi x\sin(x)dx.$$

(6) (3.5.7) Find all solutions f on $[0,1]$ and all corresponding λ to the equation:

$$f'' + \lambda f = 0, \quad f(0) = 0, \quad f'(1) = -f(1).$$

Hint: As we have computed before, consider three cases, $\lambda = 0$, $\lambda > 0$, and $\lambda < 0$. Use the boundary conditions to solve for all the possible f.

(7) (3.4.2) Find constants a, b, A, B, C such that $f_0(x) = 1$, $f_1(x) = ax + b$, and $f_2(x) = Ax^2 + Bx + C$ are an orthonormal set in $L^2_w(0,\infty)$ where $w(x) = e^{-x}$. Hint: For f_1 first you want the weighted scalar product with f_0 to be zero, so you want

$$\langle f_0, f_1\rangle_w = \int_0^\infty f_0(x)\overline{f_1(x)}e^{-x}dx = 0 = \int_0^\infty (ax+b)e^{-x}dx.$$

This will give you an equation expressing b in terms of a. Next, you also want

$$||f_1||_w^2 = 1 = \int_0^\infty (ax+b)^2 e^{-x}dx,$$

so substituting for your expression for b in terms of a, you get an equation for a. Repeat the same thing for f_2, demanding that:

$$\langle f_j, f_2\rangle_w = \int_0^\infty f_j(x)f_2(x)e^{-x}dx = 0, \quad j = 0, 1$$

and

$$||f_2||_w^2 = \int_0^\infty |f_2(x)|^2 e^{-x}dx = 1.$$

This will give you three equations for the three unknowns A, B, and C, for which you can solve!

(8) (4.2.3) Let $f(x)$ be the initial temperature at the point x in a rod of length ℓ, mathematicized as the interval $[0,\ell]$. Assume that heat is supplied at a constant rate at the right end, in particular $u_x(\ell,t) = A$ for a constant value A, and that the left end is held at the constant temperature 0, so that $u(0,t) = 0$. Find a series expansion for the temperature $u(x,t)$ such that the initial temperature is given by $f(x)$. Hint: Divide and conquer. First find a so-called steady state solution, that is find a function $g(x)$ which does not depend on t which satisfies

$$(\partial_t - \partial_{xx})g = 0, \quad g(0) = 0, \quad g'(\ell) = A.$$

Now, since g does not depend on t, when you apply the heat operator you just get

$$-g''(x) = 0, \quad g(0) = 0, \quad g'(\ell) = A.$$

Find g which solves this. Now, look for a solution u which satisfies

$$u_t - u_{xx} = 0, \quad u(0,t) = u_x(\ell,t) = 0, \quad u(x,0) = f(x) - g(x).$$

You can use the methods from last week, separation of variables, superposition (since everything including the BCs are homogeneous), and Fourier series (Hilbert spaces!) to solve for u. The full solution will then be

$$u(x,t) + g(x).$$

References

[1] Gerald B. Folland, *Fourier Analysis and Its Applications,* Pure and Applied Undergraduate Texts Volume 4, (1992).