FOURIER ANALYSIS & METHODS

JULIE ROWLETT

ABSTRACT. Caveat Emptor! These are just informal lecture notes. Errors are inevitable! Read at your own risk! Also, this is by no means a substitute for the textbook, which is warmly recommended: *Fourier Analysis and Its Applications,* by Gerald B. Folland. He was the first math teacher I had at university, and he is awesome. A brilliant writer. So, why am I even doing this? Good question...

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Recall the definition of a regular SLP:

(1) a formally self-adjoint operator

$$L(f) = (rf')' + pf,$$

where r and p are real valued, r, r', and p are continuous, and $r > 0$ on $[a, b]$.

(2) self-adjoint boundary conditions:

$$B_i(f) = \alpha_i f(a) + \alpha_i' f'(a) + \beta_i f(b) + \beta_i' f'(b) = 0, \quad i = 1, 2.$$

The self adjoint condition further requires that the coefficients $\alpha_i, \alpha_i', \beta_i, \beta_i'$ are such that for all f and g which satisfy these conditions

$$r(\bar{g}f' - \bar{g}'f)\big|_a^b = 0.$$

(3) a positive, continuous function w on $[a, b]$.

The SLP is to find all solutions to the BVP

$$L(f) + \lambda w f = 0, \quad B_i(f) = 0, \quad i = 1, 2.$$

The eigenvalues are all numbers λ for which there exists a corresponding non-zero eigenfunction f so that together they satisfy the above equation, and f satisfies the boundary condition. The magical theorem about SLPs says that for such a regular SLP, there exists solutions $\{\phi_n\}_{n\geq 1}$ with corresponding eigenvalues λ_n such that these $\{\phi_n\}_{n\geq 1}$ are an orthogonal basis for the weighted $\mathcal{L}^2$ space, $\mathcal{L}^2_w(a, b)$. Moreover, these eigenvalues are all *real*. Let's see just what makes this theorem so magical...

1.1. SLP example.

We wish to solve the following SLP:

$$(xf')' + \lambda x^{-1} f = 0, \quad f(1) = f(b) = 0, \quad b > 1.$$

In this example the function $r(x) = x$, and the function $p(x) = 0$, whilst the weight function $w(x) = x^{-1}$. Let us consider three cases for λ.

Case $\lambda = 0$: If $\lambda = 0$, then the equation becomes

$$xf'' + f' = 0,$$

which we can re-arrange to

$$\frac{f''}{f'} = -\frac{1}{x}.$$

The left side is the derivative of $\log(f')$. So, integrating both sides (saving the constant for later):

$$\log(f') = -\log(x).$$

Exponentiating both sides we get

$$f' = \frac{1}{x} \implies f(x) = A\log(x) + B,$$

for some constants A and B. The boundary conditions demand that

$$f(1) = 0 \implies B = 0.$$

The other boundary condition demands that

$$f(b) = 0 \implies A = 0, \text{ since } b > 1 \text{ so } \log(b) > 0.$$

We are left with the zero function. That is never an eigenfunction. So $\lambda = 0$ is not an eigenvalue for this SLP.

Case $\lambda > 0$: If $\lambda > 0$, we observe that the equation we have is something called an Euler equation. (Or we look up the ODE section of Beta and search for this type of ODE, and see that Beta tells us this is an Euler equaiton). Consequently, we look for solutions of the form

$$f(x) = x^\nu.$$

The differential equation we wish to solve is:

$$xf'' + f' + \lambda x^{-1} f = 0 \implies x^2 f'' + xf' + \lambda f = 0,$$

so substituting $f(x) = x^\nu$, this becomes

$$x^2(\nu)(\nu-1)x^{\nu-2} + x\nu x^{\nu-1} + \lambda x^\nu = 0.$$

This simplifies to:

$$x^\nu \left(\nu^2 - \nu + \nu + \lambda\right) = 0 \implies \nu^2 = -\lambda.$$

Since $\lambda > 0$, this means

$$\nu = \pm i\sqrt{\lambda}.$$

So, a basis of solutions is $x^{i\sqrt{|\lambda|}}$ and $x^{-i\sqrt{\lambda}}$. Note that

$$x^{\pm i\sqrt{\lambda}} = e^{\pm i\sqrt{\lambda}\log(x)}.$$

By Euler's formula, an equivalent basis of solutions is

$$\cos(\sqrt{\lambda}\log(x)), \quad \sin(\sqrt{\lambda}\log(x)).$$

Hence in this case our solution is of the form:

$$f(x) = A\cos(\sqrt{\lambda}\log(x)) + B\sin(\sqrt{\lambda}\log(x)).$$

The boundary conditions demand that

$$f(1) = 0 \implies A = 0.$$

The second boundary condition demands that

$$B\sin(\sqrt{\lambda}\log(b)) = 0.$$

Since we do not seek the zero function, we presume that $B \neq 0$ and thus require

$$\sin(\sqrt{\lambda}\log(b)) = 0 \implies \sqrt{\lambda}\log(b) = n\pi, \quad n \in \mathbb{N}.$$

We therefore have countably many eigenfunctions and eigenvalues, which we may index by the natural numbers, writing

$$\lambda_n = \frac{n^2\pi^2}{(\log b)^2}, \quad f_n(x) = \sin\left(\frac{n\pi\log(x)}{\log(b)}\right).$$

Nice.

The last case to consider is **case** $\lambda < 0$**:** We proceed similarly as above and obtain that a basis of solutions is

$$x^{\pm\sqrt{|\lambda|}}.$$

Write our solution as

$$f(x) = Ax^{\sqrt{|\lambda|}} + Bx^{-\sqrt{|\lambda|}}.$$

The boundary conditions demand that:

$$f(1) = 0 \implies A + B = 0 \implies B = -A.$$

The next boundary condition demands that:

$$f(b) = Ab^{\sqrt{|\lambda|}} - Ab^{-\sqrt{|\lambda|}} = 0 \implies A = 0 \text{ or } b^{\sqrt{|\lambda|}} = b^{-\sqrt{|\lambda|}} \implies b^{2\sqrt{|\lambda|}} = 1 \implies \sqrt{|\lambda|} = 0 ↯.$$

Thus the only way for the boundary conditions to be satisfied is if the eigenfunction is the zero function, but this is not an eigenfunction! Hence no negative λ solutions.

The magical SLP theorem tells us that these rather peculiar functions

$$\{f_n(x)\}_{n\geq 1}$$

are an orthogonal basis for $\mathcal{L}^2_{1/x}(1, b)$. This means that for any $g \in \mathcal{L}^2_{1/x}(1, b)$, we can expand it as a Fourier series with respect to this basis. The coefficients will be

$$\frac{\langle g, f_n\rangle_{1/x}}{||f_n||^2_{1/x}}, \quad \langle g, f_n\rangle_{1/x} = \int_1^b g(x)\overline{f_n(x)}x^{-1}dx, \quad ||f_n||^2_{1/x} = \int_1^b |f_n(x)|^2 x^{-1}dx.$$

If the function we wish to expand is specified, we could compute these integrals.

1.1.1. *SLP example for a PDE.* Here is how the SLP theory can be useful in practice. We are given the problem

$$u_t - u_{xx} = 0, \quad u_x(0, t) = \alpha u(0, t), \quad u_x(l, t) = -\alpha u(l, t), \quad u(x, 0) = f(x).$$

Above, we assume that

$$\alpha > 0, \quad f \in \mathcal{L}^2.$$

These BCs come from Newton's law of cooling: the temperature gradient across the ends is proportional to the temperature difference between the ends and the surrounding medium. It is a homogeneous PDE, so we have good chances of being able to solve it using separation of variables. Thus, we write

$$u(x, t) = X(x)T(t) \implies T'(t)X(x) - X''(x)T(t) = 0 \implies \frac{T'}{T} = \frac{X''}{X}.$$

This means both sides are equal to a constant. Call it λ. We start with the x side, because we have more information about that due to the BCs. Are they self-adjoint BCs? Let's check! In the definition of SLP, we are looking for X to satisfy

$$\frac{X''}{X} = \lambda \iff X'' = \lambda X \iff X'' - \lambda X = 0.$$

OBS! The relationship between the constant we have named λ from the PDE has the *opposite* sign as the corresponding term in an SLP. So, the SLP would look like

$$X'' + \Lambda X = 0 \quad \Lambda = -\lambda.$$

The r and w are both 1 in the definition of SLP, and the p is 0. The $a = 0$ and $b = l$. So, we need to check that if f and g satisfy

$$f'(0) = \alpha f(0), \quad g'(l) = -\alpha g(l)$$

then

$$(\bar{g} f' - \bar{g}' f)\big|_0^l = 0.$$

We plug it in

$$\bar{g}(l) f'(l) - \bar{g}'(l) f(l) - \bar{g}(0) f'(0) + \bar{g}'(0) f(0)$$
$$= -\bar{g}(l)\alpha f(l) + \alpha \bar{g}(l) f(l) - \bar{g}(0)\alpha f(0) + \alpha \bar{g}(0) f(0) = 0.$$

Yes, the BC is a self-adjoint BC. So, the SLP theorem says there exists an $\mathcal{L}^2$ ONB of eigenfunctions. What are they? We check the cases.

$$X'' = \lambda X.$$

What if $\lambda = 0$? Then

$$X(x) = ax + b.$$

To get

$$X'(0) = \alpha X(0) \implies a = \alpha b \implies b = \frac{a}{\alpha}.$$

Next,

$$X'(l) = -\alpha X(l) \implies a = -\alpha \left(al + \frac{a}{\alpha} \right) = -a(\alpha l + 1).$$

Presumably $a \neq 0$ because if $a = 0$ the whole solution is just 0. So, we can divide by it and we get

$$\implies 1 = -(\alpha l + 1) \implies \alpha l = -2.$$

Since $l > 0$ and $\alpha > 0$, this is impossible. So, no solutions for $\lambda = 0$.

Next we try $\lambda > 0$. Then the solution looks like

$$X(x) = ae^{\sqrt{\lambda}x} + be^{-\sqrt{\lambda}x}$$

or equivalently, we can use sinh and cosh, to write

$$X(x) = a\cosh(\sqrt{\lambda}x) + b\sinh(\sqrt{\lambda}x).$$

We try out the BCs. They require

$$X'(0) = \alpha X(0) \iff a\sqrt{\lambda}\sinh(0) + b\sqrt{\lambda}\cosh(0) = \alpha\left(a\cosh(0) + b\sinh(0)\right)$$
$$\iff b\sqrt{\lambda} = \alpha a \implies b = \frac{\alpha a}{\sqrt{\lambda}}.$$

We check out the other BC:

$$X'(l) = -\alpha X(l) \iff a\sqrt{\lambda}\sinh(\sqrt{\lambda}l) + \alpha a\cosh(\sqrt{\lambda}l) = -\alpha\left(a\cosh(\sqrt{\lambda}l) + \frac{\alpha a}{\sqrt{\lambda}}\sinh(\sqrt{\lambda}l)\right).$$

$$\iff a\sqrt{\lambda}\sinh(\sqrt{\lambda}l) + \frac{\alpha^2 a}{\sqrt{\lambda}}\sinh(\sqrt{\lambda}l) = -2\alpha a\cosh(\sqrt{\lambda}l)$$

If $a = 0$ the whole solution is zero, so we presume that is not the case and divide by a. Then this requires

$$\frac{\sinh(\sqrt{\lambda}l)}{\cosh(\sqrt{\lambda}l)} = \frac{-2\alpha}{\sqrt{\lambda} + \alpha^2/\sqrt{\lambda}}.$$

Equivalently

$$\tanh(\sqrt{\lambda}l) = \frac{-2\alpha\sqrt{\lambda}}{\lambda+\alpha^2}.$$

Are there solutions to this equation? Well, $\sqrt{\lambda}l > 0$. So the left side is positive, but the right side is negative. So, this equation has no solutions.

Thus, we finally try $\lambda < 0$. Then the solution looks like

$$X(x) = a\cos(\sqrt{|\lambda|}x) + b\sin(\sqrt{|\lambda|}x).$$

To get

$$X'(0) = \alpha X(0) \implies b\sqrt{|\lambda|} = \alpha a \implies b = \frac{\alpha a}{\sqrt{|\lambda|}}.$$

Next we need

$$X'(l) = -\alpha X(l)$$

$$\implies -a\sqrt{|\lambda|}\sin(\sqrt{|\lambda|}l) + \frac{\alpha a}{\sqrt{|\lambda|}}\sqrt{|\lambda|}\cos(\sqrt{|\lambda|}l) = -\alpha\left(a\cos(\sqrt{|\lambda|}l) + \frac{\alpha a}{\sqrt{|\lambda|}}\sin(\sqrt{|\lambda|}l)\right).$$

Presumably $a \neq 0$ because if that is the case then the whole solution is 0. So, we may divide by a, and we need

$$2\alpha\cos\sqrt{|\lambda|} = \sin(\sqrt{|\lambda|}l)\left(\sqrt{|\lambda|} - \frac{\alpha^2}{\sqrt{|\lambda|}}\right).$$

This is equivalent to

$$\frac{2\alpha}{\sqrt{|\lambda|} - \frac{\alpha^2}{\sqrt{|\lambda|}}} = \tan(\sqrt{|\lambda|}l)$$

$$\iff \frac{2\alpha\sqrt{|\lambda|}}{|\lambda| - \alpha^2} = \tan(\sqrt{|\lambda|}l).$$

Well, that's pretty weird, but according to the SLP theory, the sequence

$$\{\lambda_n\}_{n\geq 1} \text{ and } \{X_n(x)\}_{n\geq 1}, \quad X_n(x) = a_n\left(\cos(\sqrt{|\lambda_n|}x) + \frac{\alpha}{\sqrt{|\lambda_n|}}\sin(\sqrt{|\lambda_n|}x)\right)$$

of eigenvalues and corresponding eigenfunctions is an orthogonal basis of $\mathcal{L}^2$. Here since we are solving a PDE, it is most convenient to leave the coefficients simply as a_n and solve for them according to the initial conditions of the PDE.

The partner functions

$$T_n(t) \text{ satisfy } T_n'(t) = \lambda_n T_n(t) \implies T_n(t) = e^{\lambda_n t}.$$

Here it is good to note that the $\lambda_n < 0$ and the tend to $-\infty$ as $n \to \infty$ which follows from the big magical theorem on SLPs, because in the SLP terminology,

$$\Lambda_n = -\lambda_n \to \infty \implies \lambda_n \to -\infty.$$

So, for heat, that is realistic. We build the solution using superposition because the PDE is linear and homogeneous, so

$$u(x,t) = \sum_{n\geq 1} T_n(t)X_n(x).$$

Since we wish this to be equal to the initial data at $t = 0$, we demand

$$u(x, 0) = \sum_{n \geq 1} a_n \left(\cos(\sqrt{|\lambda_n|}x) + \frac{\alpha}{\sqrt{|\lambda_n|}} \sin(\sqrt{|\lambda_n|}x) \right) = f(x).$$

By the SLP theory, the functions above form an OB, so we can expand our initial data function in terms of this OB. To do this we compute

$$a_n = \frac{\langle f(x), \cos(\sqrt{|\lambda_n|}x) + \frac{\alpha}{\sqrt{|\lambda_n|}} \sin(\sqrt{|\lambda_n|}x) \rangle}{|| \cos(\sqrt{|\lambda_n|}x) + \frac{\alpha}{\sqrt{|\lambda_n|}} \sin(\sqrt{|\lambda_n|}x)||^2},$$

where

$$\langle f(x), \cos(\sqrt{|\lambda_n|}x) + \frac{\alpha}{\sqrt{|\lambda_n|}} \sin(\sqrt{|\lambda_n|}x) \rangle = \int_0^l f(x) \overline{(\cos(\sqrt{|\lambda_n|}x) + \frac{\alpha}{\sqrt{|\lambda_n|}} \sin(\sqrt{|\lambda_n|}x))} dx,$$

$$|| \cos(\sqrt{|\lambda_n|}x) + \frac{\alpha}{\sqrt{|\lambda_n|}} \sin(\sqrt{|\lambda_n|}x)||^2 = \int_0^l | \cos(\sqrt{|\lambda_n|}x) + \frac{\alpha}{\sqrt{|\lambda_n|}} \sin(\sqrt{|\lambda_n|}x)|^2 dx.$$

1.2. **Another SLP example.** Consider the problem

$$(x^2 f')' + \lambda f = 0, \quad f(1) = f(b) = 0, \quad b > 1.$$

Here we have $r(x) = x^2$ and $w(x) = 1$. The equation is:

$$2xf' + x^2 f'' + \lambda f = 0.$$

We shall consider the three cases for λ.

Case $\lambda = 0$: In this case the equation simplifies to

$$x^2 f'' + 2xf' = 0 \implies \frac{f''}{f'} = -\frac{2}{x} \implies (\log(f'))' = -\frac{2}{x} \implies \log(f') = -2 \log x \implies f' = e^{-2 \log x} = x^{-2}.$$

So, this gives us a solution of the form

$$f(x) = -A\frac{1}{x} + B.$$

Let us verify the boundary conditions. We require $f(1) = 0$ so this means

$$-A + B = 0 \implies B = A.$$

We also require $f(b) = 0$ so this means

$$-A\frac{1}{b} + B = 0 = \frac{-A}{b} + A \implies \frac{A}{b} = A \implies b = 1 \text{ or } A = 0.$$

So since $b > 1$ the only solution here is the zero function which is not an eigenfunction.

Case $\lambda > 0$: We consider the fact that this is an Euler equation, so we look for solutions of the form $f(x) = x^\nu$. Then the equation looks like:

$$x^2(\nu)(\nu - 1)x^{\nu-2} + 2x(\nu)x^{\nu-1} + \lambda x^\nu = 0 \iff x^\nu \left(\nu^2 - \nu + 2\nu + \lambda\right) = 0$$

so we need ν to satisfy:

$$\nu^2 + \nu + \lambda = 0.$$

This is a quadratic equation, so we solve it:

$$\nu = -\frac{1}{2} \pm \sqrt{\frac{1}{4} - \lambda}.$$

So, actually the cases $\lambda > 0$ and $\lambda < 0$ really should split up into whether $\lambda = \frac{1}{4}$ or is larger or smaller. If $\lambda = \frac{1}{4}$, then we are only getting one solution this way, $x^{-1/2}$. To get a second solution we multiply by $\log x$.

Exercise 1. *Plug the function $x^{-1/2} \log x$ into the SLP for the value $\lambda = \frac{1}{4}$. Verify that it satisfy the equation.*

Now, let's see if such a function will satisfy the boundary conditions. We need

$$Ax^{-1/2} + Bx^{-1/2} \log(x)\Big|_{x=1} = 0 \implies A = 0.$$

We also need

$$Bb^{-1/2} \log(b) = 0, \quad b > 1 \implies B = 0.$$

So we only get the zero solution in this case.

When $\lambda < \frac{1}{4}$, solutions are of the form

$$Ax^{\nu_+} + Bx^{\nu_-}, \quad \nu_\pm = -\frac{1}{2} \pm \sqrt{\frac{1}{4} - \lambda}.$$

Exercise 2. *Check the boundary conditions. Verify that they are satisfied if and only if $A = B = 0$.*

Finally we consider $\lambda > \frac{1}{4}$. Then we have

$$\nu_\pm = -\frac{1}{2} \pm i\sqrt{\lambda - \frac{1}{4}} \implies f(x) = \frac{A}{\sqrt{x}} x^{i\sqrt{\lambda - 1/4}} + \frac{B}{\sqrt{x}} x^{-i\sqrt{\lambda-1/4}}.$$

Using Euler's formula, this is equivalently expressed as

$$\frac{\alpha}{\sqrt{x}} \cos(\sqrt{\lambda - 1/4} \log x) + \frac{\beta}{\sqrt{x}} \sin(\sqrt{\lambda - 1/4} \log x).$$

Due to the boundary condition at $x = 1$ we must have $\alpha = 0$. So to obtain the other boundary condition, we need

$$\sin(\sqrt{\lambda - 1/4} \log b) = 0 \implies \sqrt{\lambda - 1/4} \log b = n\pi, \quad n \in \mathbb{N}.$$

Hence

$$\lambda = \lambda_n = \frac{1}{4} + \frac{n^2\pi^2}{(\log b)^2}, \quad f_n(x) = x^{-1/2} \sin\left(\frac{n\pi \log x}{\log b}\right).$$

Note that in general we are not bothering to normalize our eigenfunctions because it is rather tedious and not fundamental to our learning experience in this subject.