FOURIER ANALYSIS PREP!

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1. Welcome to Fourier Analysis!

To make sure that you're prepared for this course, we've put together a few questions. (This is mathspeak, and we=I). The purpose is to help you check whether you are up to speed with the pre-requisites you'll need to stay abreast in this course. If you have any questions about what to review, please let me (Julie) know! I am always happy to help! As with all mathematics, understanding the concepts and how to work with them is crucial. So, if you are not understanding something, please ask! It is equally important that you solve problems as well, because doing math is the best way to learn math. However, sometimes when we're trying to solve problems, we get stuck. (Even I get stuck!) It is good to struggle with problems for a while after getting stuck. It might not seem like this time struggling has any benefit to you whatsoever, but trust me, it does. It is like exercise for your math brain. So, it's good to struggle for a while at least before going and asking for help. When is it time to ask for help then? In my experience doing math, if you have really, genuinely worked on a problem and struggled with it, then you will be able to state the problem, in its entirety, from memory. This will happen *without you sitting down and memorizing the problem directly.* In other words, the process of struggling with the problem will burn that problem into your brain so that it is completely memorized. If you struggle long enough so that you get to this point, it is totally fine to STOP. Relax. Come ask me (or any of the friendly and knowledgeable teaching assistants) for help! We are here to help! Don't leave the problem unsolved, make sure you come and get help on it. This way it won't come back to haunt you in the exam.

With this little introduction, welcome! Let the mathematical adventure begin!

2. Linear Algebra

(1) Give an example of an orthonormal basis for $\mathbb{R}^3$ which is *not* the standard one.

(2) Let's call your basis vectors now u, v, and w. Write the vector

$$\begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix} = au + bv + cw,$$

for $a, b, c \in \mathbb{R}$.

(3) Let $x, y \in \mathbb{R}^n$ for some $n \geq 2$ be non-zero vectors. What can you say about x and y, geometrically, if you know that

$$x \cdot y = 0?$$

(4) What geometric information does
$$\sqrt{x \cdot x}$$
tell you about the vector, x?

3. Complex numbers

Complex numbers are your friends! Rather than making things more complex, they often make things simpler. For this reason, let's recall some basics.

(1) A complex number looks like
$$a + bi,$$
where a and b are real numbers, and i is a thing defined so that
$$i^2 = -1.$$
Here, a is called the real part, and b is called the imaginary part. Note that both the real part and the imaginary part are real numbers(!) It's just that the imaginary part has the i attached to it. Write down some examples, like $1 + i$, $3 - 2i$.

(2) Every real number is also a complex number because 0 is real, so if $a \in \mathbb{R}$ is real, then
$$a = a + 0i.$$

(3) Recall Euler's formula:
$$e^{ix} = \cos(x) + i \sin(x),$$
which holds for all $x \in \mathbb{R}$. This formula is SUPER IMPORTANT! A complex number whose real part is zero, but whose imaginary part is non-zero is called pure imaginary. If z is pure imaginary, what can you say about
$$e^{iz}?$$

(4) Complex conjugate is, for $z = a + ib$,
$$\overline{z} = a - ib.$$
What do you get from
$$z + \overline{z}?$$

(5) The real part of z is often denoted
$$\Re(z)$$
which for $z = a + ib$ is a. The imaginary part is denoted
$$\Im(z),$$
which for $z = a + ib$ is b. Show that
$$\frac{z - \overline{z}}{i} = 2\Im z.$$

(6) The modulus of a complex number (or equivalently, the absolute value) is defined to be
$$|z| := \sqrt{z\overline{z}}.$$
For $z = a + ib$ show that
$$|z| = \sqrt{a^2 + b^2}.$$

What geometric information does $|z|$ tell us about z sitting in the complex plane?

(7) Every complex number can be written as

$$z = re^{i\theta} = r\cos\theta + ir\sin\theta, \quad r = |z|.$$

What geometric information do r and θ tell us about where z is sitting in the complex plane?

(8) Show that if $|z| = 1$ then

$$\overline{z} = \frac{1}{z}.$$

(9) Show that if

$$z = e^{i\theta}$$

for some $\theta \in \mathbb{R}$, then

$$|z| = 1.$$

(10) * We say that z is an n^{th} root of unity if

$$z^n = 1.$$

Here we are assuming n is a natural number (usually, we roll American style with the natural numbers, in that we start with one not with zero). Natural numbers are denoted by $\mathbb{N}$. Prove that for each n there are precisely n distinct complex numbers which are n^{th} roots of unity. Prove that if you draw these in the complex plane, then they form the vertices of a regular n-gon.

4. Analysis

(1) Determine whether or not these converge, and in case of convergence, compute the limit

$$\lim_{x\to 0} x\tan\left(\frac{1}{x}\right)$$

$$\lim_{x\to\frac{\pi}{2}} \frac{e^x - \sin(x)}{x - \frac{\pi}{2}}$$

$$\lim_{x\to 0} \frac{\tan(5x)}{x}$$

$$\lim_{x\to 0} x\sin\left(\frac{1}{x}\right)$$

$$\lim_{x\to\infty} \frac{6x^8 + e^x}{3^{2x} + 500x}$$

$$\sum_{n=0}^{\infty} \frac{5^n}{n!}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

(2) Compute:

$$\partial_x(\sin(e^{x-y})), \quad \partial_y(\sin(e^{x-y})), \quad \partial_t(\sin(e^{x-y})).$$

(3) Assume that f depends on a single variable called t which is *different and independent from* a variable called z. Assume that g depends on this variable called z (and *only* on this variable, z). If

$$f(x) = g(z),$$

then what can you conclude about f and g?

(4) Compute:

$$\int_0^2 xe^{inx}dx, \qquad \text{for } n \in \mathbb{Z}.$$

(5) Compute:

$$\int_0^L \sin^2(x)dx.$$

(6) What does it really mean to write

$$\int_{\mathbb{R}} f(x)dx$$

or

$$\int_0^\infty f(x)dx?$$

(7) Compute:

$$\int_0^\infty \frac{x}{e^x}dx.$$

(8) * Compute:

$$\int_{\mathbb{R}} \frac{\sin(x)}{x}dx.$$

5. Mathematical logic and general mathematical thinking

If you would like to brush up on mathematical logic, mathematical thinking, and mathematical proofs, you may enjoy checking out my book, *Blast into Math!* It's free! It's got cute illustrations too! Check it out here:
`https://bookboon.com/se/blast-into-math-ebook`
You can read in my book about the different number systems, $\mathbb{N}$, $\mathbb{Z}$ (why on earth do we need negative numbers? read and find out!), $\mathbb{Q}$, $\mathbb{R}$ (what the heck are real numbers which are irrational? read and find out!), as well as $\mathbb{C}$. The book also explains the concepts:

- limit of a sequence (and what is a sequence anyways)?
- limit of a series (and what is a series anyways)?

So, if you are a bit shaky with any of these concepts, or if you are just curious, please check it out!

Always remember, if you need any help, please just ask!