Fourieranalys MVE030 och Fourier Metoder MVE290 22.mars.2019
Betygsgränser: 3: 40 poäng, 4: 53 poäng, 5: 67 poäng.
Maximalt antal poäng: 80.
Hjälpmedel: BETA.
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1. Lös problemet:

$$\begin{cases} u(0,t) = 0 & t > 0 \\ u_t(x,t) - u_{xx}(x,t) = 0 & t, x > 0 \\ u(x,0) = f(x) \in \mathcal{L}^2((0,\infty)) \cap \mathcal{C}^0((0,\infty)) & x > 0 \end{cases}$$

(10 p)

Well, this is a PDE in a half space. To figure out what we should do, let's investigate the boundary condition. The boundary condition is that

$$u(0,t) = 0.$$

This is rather nice. To achieve such a condition, as we have seen in examples and exercises, we should extend the initial data f oddly. Moreover, we also see that $f \in \mathcal{L}^2$, which indicates that Fourier transform methods have good odds of working. We know that the Fourier transform plays nicely with extending evenly and oddly, in the sense that the Fourier transform preserves these properties: Fourier transform an even function, the result is even; Fourier transform an odd function, the result is odd. On the other hand, the Fourier transform does *not* play nicely by say extending to be identically zero on the negative real line. If you extend this way, then apply the Fourier transform, the result will *not* necessarily be zero on the negative real line.

So, all these considerations tell us to extend f evenly or oddly, and due to the condition $u(0,t) = 0$, we shall extend oddly. (Just think about sine and cosine, the sine is the odd one, and it is the one who vanishes at zero).

Let

$$f_o(x) = f(x), \quad x > 0, \quad f_o(x) = -f(-x), \quad x < 0.$$

Then let's apply the Fourier transform to the PDE in the x variable:

$$\hat{u}_t(\xi,t) - \widehat{u_{xx}}(\xi,t) = 0.$$

The properties of the Fourier transform (so generously given to us at the end of this exam) say that

$$\widehat{u_{xx}}(\xi,t) = (-i\xi)^2 \hat{u}(\xi,t),$$

so our equation becomes

$$\hat{u}_t(\xi,t) + \xi^2 \hat{u}(\xi,t) = 0 \implies \hat{u}(\xi,t) = a(\xi)e^{-\xi^2 t}.$$

(Above we have solved the ODE for the Fourier transform where the ODE variable is the variable t, and the variable ξ is an independent variable). The initial condition is that

$$\hat{u}(\xi,0) = a(\xi) = \hat{f}_o(\xi).$$

So,

$$\hat{u}(\xi,t) = \hat{f}_o(\xi)e^{-\xi^2 t}.$$

Well, the Fourier transform sends a convolution to a product. We look at the table to find a function whose Fourier transform is $e^{-\xi^2 t}$. We know a function whose Fourier transform is $\hat{f}_o(\xi)$, simply f_o. So,

$$u(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} f_o(y) e^{-(x-y)^2/(4t)} dy.$$

To put this in terms of the original function, and verify the boundary condition, we recall the definition of f_o as being an odd extension, so

$$u(x,t) = \frac{1}{\sqrt{4\pi t}} \left(\int_{-\infty}^{0} f_o(y) e^{-(x-y)^2/(4t} dy + \int_0^\infty f(y) e^{-(x-y)^2/(4t)} dy \right).$$

We can turn the integral on the negative real axis into an integral on the positive real axis. To do this, let $z = -y$, then

$$\int_{-\infty}^{0} f_o(y) e^{-(x-y)^2/(4t} dy = \int_{\infty}^{0} f_o(-z) e^{-(x+z)^2/(4t)} (-dz) = \int_0^\infty f_o(-z) e^{-(x+z)^2/(4t)} dz.$$

Since

$$f_o(-z) = -f_o(z) \quad z > 0,$$

this is

$$-\int_0^\infty f(z)e^{-(x+z)^2/(4t)}dz.$$

Now, the name of the variable of integration is irrelevant, so we may as well re-name it back to y, and then we have

$$u(x,t) = \frac{1}{\sqrt{4\pi t}} \int_0^\infty f(y) \left(e^{-(x-y)^2/(4t)} - e^{-(x+y)^2/(4t)} \right) dy.$$

If $x = 0$ then the two terms in parentheses cancel. So we see that the boundary condition is satisfied. Since we worked always with $\mathcal{L}^2$ functions, the convolution approximation theorem guarantees that the initial condition is also satisfied.

Since it might be helpful, here is basically how partial credit will be dished out. In case any of these items is somewhat messed up, but half-right, you'd get 1p instead of 2p.

(a) (2p) Choosing to use Fourier transform methods.

(b) (2p) Choosing to extend the initial condition oddly.

(c) (2p) Correctly Fourier transforming the PDE.

(d) (2p) Solving the ODE for the Fourier transform of the solution correctly.

(e) (2p) Correctly inverting the Fourier transform to obtain the solution (going backwards correctly).

2. Lös problemet:

$$\begin{cases} u(0,t) = e^t & t > 0 \\ u_t(x,t) - u_{xx}(x,t) = 0 & t, x > 0 \\ u(x,0) = 0 & x > 0 \end{cases}$$

(10 p)

This problem has different features. Specifically the boundary condition:

$$u(0,t) = e^t.$$

Moreover, the initial condition is zero. With the Fourier transform method, we are usually getting some convolution type stuff involving the initial data. If we were to obtain something like that here, it would just vanish since the initial data is zero. If we were to try Fourier transform methods in the t variable, it would fail miserably because e^t is very much not Fourier transformable.

So, this indicates that a different approach is required. In particular, all of these considerations suggest using the Laplace transform in the t variable. We Laplace transform the PDE in the t variable:

$$\widetilde{u_t}(x,z) - \widetilde{u}_{xx}(x,z) = 0.$$

We use the properties of the Laplace transform and the nice homogeneous initial condition to obtain:

$$z\widetilde{u}(x,z) - \widetilde{u}_{xx}(x,z) = 0.$$

We solve this ODE to obtain:

$$\widetilde{u}(x,z) = a(z)e^{-x\sqrt{z}} + b(z)e^{x\sqrt{z}}.$$

The properties of the Laplace transform imply (indeed it was a Theorem) that anything which is Laplace-transformable will $\to 0$ as the real part of z tends to infinity. For $x > 0$ (which it is since we work in the positive real line on this problem) the second term will not satisfy that unless b has some really great decay properties. However b doesn't depend on x so if $x \to \infty$ also, then b cannot save this term from growing exponentially. Thus, we try to solve the problem using only the other term. The boundary condition says:

$$\widetilde{u}(0,z) = \widetilde{e^t}(z) = a(z) \implies \widetilde{u}(x,z) = \widetilde{e^t}(z)e^{-x\sqrt{z}}.$$

Now, we could compute the Laplace transform of e^t, it is

$$\int_0^\infty e^t e^{-tz} dt = \left.\frac{e^{t(1-z)}}{1-z}\right|_0^\infty = \frac{1}{z-1}.$$

So, this is fine for real part of z greater than one. That is the usual property of the Laplace transform.

We know that the Laplace transform takes a convolution to a product. We know where the first term came from, so we look for a function whose Laplace transform is $e^{-x\sqrt{z}}$. We look at the lovely table. We see that to get $2a^{-1}\sqrt{\pi}e^{-a\sqrt{z}}$ as the Laplace transform we should start with $H(t)t^{-3/2}e^{-a^2/(4t)}$. So with our problem, we would want $a = x$, and to obtain $e^{-x\sqrt{z}}$ as the Laplace transform we should start with

$$\frac{x}{2\sqrt{\pi}t^{3/2}}H(t)e^{-x^2/(4t)}.$$

Hence

$$u(x,t) = \int_{\mathbb{R}} H(s)e^s H(t-s)\frac{x}{2\sqrt{\pi}(t-s)^{3/2}}e^{-x^2/(4(t-s))}ds.$$

This is because the Laplace transform is in the t variable, so that's the variable for the convolution, and also because the Laplace transform needs the functions inside to be zero for negative values (hence the Heavyside factors). With these Heavyside factors in mind, we obtain

$$u(x,t) = \int_0^t e^s \frac{x}{2\sqrt{\pi}(t-s)^{3/2}}e^{-x^2/(4(t-s))}ds.$$

Since it might be helpful, here is basically how partial credit will be dished out. In case any of these items is somewhat messed up, but half-right, you'd get 1p instead of 2p.

(a) (2p) Choosing to use Laplace transform methods.

(b) (2p) Correctly Laplace transforming the PDE.

(c) (2p) Solving the ODE for the Laplace transform of the solution correctly to get the general solution.

(d) (2p) Discarding the non-Laplace-transformable part of the solution and using the BC to determine the Laplace transform of the solution to the PDE. (Basically going from the general solution of the ODE to the particular solution correctly here).

(e) (2p) Correctly inverting the Laplace transform to obtain the solution (going backwards correctly).

3. Lös ekvationen:

$$u(t) + \int_{-\infty}^{\infty} e^{-|t-\tau|}u(\tau)d\tau = e^{-|t|}.$$

(10p)

We have seen such equations in the exercises. The second term is a convolution, and the term on the right is one of the items on our list of Fourier transforms. So let us transform this entire equation:

$$\hat{u}(\xi) + \hat{u}(\xi)\frac{2}{\xi^2+1} = \frac{2}{\xi^2+1}.$$

This is because the Fourier transform of a convolution is the product of the Fourier transforms, and the Fourier transform of $e^{-a|x|}$ is given in the table. In our cases on both the left and right sides $a = 1$. So we solve this equation for $\hat{u}(\xi)$:

$$\hat{u}(\xi)\left(1+\frac{2}{\xi^2+1}\right) = \frac{2}{\xi^2+1} \implies \hat{u}(\xi)\left(\xi^2+3\right) = 2 \implies \hat{u}(\xi) = \frac{2}{\xi^2+3}.$$

Now, we see that to obtain such a Fourier transform, writing it like

$$\frac{1}{\sqrt{3}}\frac{2(\sqrt{3})}{\xi^2+(\sqrt{3})^2},$$

the function we ought to start with is

$$\frac{1}{\sqrt{3}}e^{-\sqrt{3}|x|}.$$

Points:

(a) (2p) Choosing to use Fourier transform methods.
(b) (2p) Correctly Fourier transforming the equation.
(c) (3p) Correctly solving for the Fourier transform of u.
(d) (3p) Inverting the Fourier transform to obtain u.

4. Lös problemet:

$$\begin{cases} u_{tt}(x,t) - u_{xx}(x,t) = e^x & 0 < t,\ 0 < x < 1 \\ u(x,0) = g(x) \in \mathcal{C}^0[0,1] & x \in [0,1] \\ u_t(x,0) = h(x) \in \mathcal{C}^0[0,1] & x \in [0,1] \\ u(0,t) = 0 = u(1,t) & t > 0 \end{cases}$$

(Antag att $g(0) = g(1) = 0$.)

(10p)

Now we have entered the geometric realm of bounded intervals. Indeed $0 < x < 1$. The boundary conditions are fantastic. The initial conditions are fine. The only issue is that the PDE is not homogeneous. However, it is *time independent*. So we can attempt to deal with this by finding a steady state (that means time independent) solution. So we first seek a function ϕ which satisfies

$$-\phi''(x) = e^x.$$

We would also like to preserve the beautiful boundary conditions, so we politely request that

$$\phi(0) = \phi(1) = 0.$$

Now the function $-e^x$ will certainly satisfy this ODE. Solutions to the homogeneous version of this ODE are linear functions. So a general solution is

$$\phi(x) = -e^x + ax + b,$$

for some constants a and b. To achieve the boundary condition at zero, we need $b = 1$. To achieve the boundary condition at 1 we need

$$0 = -e + a + 1 \implies a = e - 1.$$

So we define

$$\phi(x) = -e^x + (e-1)x + 1.$$

Now, we just need to solve a nicer problem:

$$\begin{cases} v_{tt}(x,t) - v_{xx}(x,t) = 0 & 0 < t,\ 0 < x < 1 \\ v(x,0) = g(x) - \phi(x) & x \in [0,1] \\ v_t(x,0) = h(x) \in \mathcal{C}^0[0,1] & x \in [0,1] \\ v0,t) = 0 = v(1,t) & t > 0 \end{cases}.$$

Then, the full solution will be

$$u(x,t) = \phi(x) + v(x,t).$$

Note that our initial data is still beautiful, continuous, and certainly therefore in $\mathcal{L}^2(0,1)$. Moreover, the boundary conditions are fantastic (self adjoint in particular). So Fourier series methods ought to work here.

We approach the problem at hand now by separating variables writing

$$v = X(x)T(t).$$

We put this into the PDE:

$$T''(t)X(x) - X''(x)T(t) = 0.$$

We tidy it up so that all time dependent terms are on one side, and all space dependent terms are on the other side. So, to achieve this we first divide by XT and then re-arrange:

$$\frac{T''}{T} = \frac{X''}{X}.$$

Since the two sides depend on different variables, they must both be constant. So, we look for solutions to

$$\frac{T''}{T} = \text{ constant } = \frac{X''}{X}.$$

We start with the X side because its conditions are homogeneous and simple. In particular, we seek to solve

$$X'' = \lambda X, \quad X(0) = X(1) = 0.$$

If you recognize the solutions will be sines, you can "skip to the good bit." Otherwise one needs to check all cases. First case, $\lambda = 0$. Then X would be a linear function. Linear functions cannot go up and then down. They either go up, down, or lie flat. In this case, to have $X(0) = X(1) = 0$, we need the flatline zero linear function. That will not contribute anything non-zero to our solution.

In the next case $\lambda > 0$. So, the solution to the equation could be written as either a linear combination of $e^{\pm\sqrt{\lambda}x}$ or as a linear combination of hyperbolic sine and cosine. Let us use the latter, because 0 is in our interval. Writing

$$a\cosh(\sqrt{\lambda}x) + b\sinh(\sqrt{\lambda}x)$$

the condition to vanish at $x = 0$ requires that $a = 0$. The condition to vanish at $x = 1$ would require (if we want $b \neq 0$) that $\sinh(\sqrt{\lambda}) = 0$. The only real number at which the sinh vanishes is at zero. So we would need $\lambda = 0$. However that contradicts the case we are in. Therefore the case $\lambda > 0$ yields no non-zero solutions.

Finally, we have the case $\lambda < 0$. In this case the solutions are linear combinations of $\sin(\sqrt{|\lambda|}x)$ and $\cos(\sqrt{|\lambda|}x)$. The condition to vanish at zero means that there cannot be a cosine term. Moreover, the condition to vanish at $x = 1$ means that we need $\sqrt{|\lambda|}$ to be an integer multiple of π. Consequently, all solutions we find in this way are, up to constant factors,

$$X_n(x) = \sin(n\pi x), \quad \lambda_n = -n^2\pi^2.$$

This informs us what the T function must be since

$$\frac{T_n''}{T_n} = \lambda_n = -n^2\pi^2 \implies T_n(t) = \text{ a linear combination of } \sin(n\pi t) \text{ and } \cos(n\pi t).$$

In the last step, we put together all the $X_n T_n$ pairs, by the superposition principle, because the PDE is homogeneous, thereby creating our super solution:

$$v(x,t) = \sum_{n \geq 1} X_n(x) \left(a_n \cos(n\pi t) + b_n \sin(n\pi t) \right).$$

We shall need the constant factors now to guarantee that the initial conditions are satisfied. First we have the condition at $t = 0$ for the function,

$$v(x,0) = \sum_{n \geq 1} a_n X_n(x) = g(x) - \phi(x) \implies a_n = \frac{\int_0^1 (g - \phi)\overline{X_n}}{\int_0^1 |X_n|^2}.$$

The reason we can expand the function $g(x) - \phi(x)$ in a Fourier X_n series is that the SLP theory guarantees that the functions X_n form an orthogonal basis for $\mathcal{L}^2$ on the interval $[0,1]$. Moreover, the functions g and ϕ are continuous on the closed interval, hence bounded on that interval, hence certainly elements of the Hilbert space $\mathcal{L}^2([0,1])$. So they can indeed be expanded in terms of the functions X_n.

Next we have the condition for the derivative at zero, so

$$v_t(x,0) = \sum_{n\geq 1} b_n(n\pi)X_n(x) = h(x) \implies b_n = \frac{\int_0^1 h\overline{X}_n}{n\pi\int_0^1 |X_n|^2}.$$

Similar considerations justify the expansion of h in a Fourier X_n series. We have therefore specified all quantities in our solution.

Points:

(a) (1p) Choosing to find a steady state solution to deal with the inhomogeneity in the PDE.

(b) (2p) Correctly solving for the steady state solution to solve the inhomogeneous PDE and not screw up the nice BC.

(c) (1p) Setting up the next problem to solve correctly. (homog. PDE, modified IC, same BC, then observe full solution will be sum of these two).

(d) (2p) Choosing to use separation of variables.

(e) (2p) Obtaining the X_n part of the solution correctly.

(f) (2p) Obtaining the T_n part of the solution, in particular getting the a_n and the b_n coefficients correctly.

5. Beräkna:

$$\sum_{n\geq 1} \frac{1}{\pi^2+n^2}.$$

(Tips: beräkna Fourier-serien av $e^{\pi x}$.)

(10p)

Okay, let's follow the hint and compute the Fourier series. We compute the coefficients

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{\pi x}e^{-inx}dx = \frac{1}{2\pi}\frac{e^{x(\pi-in)}}{\pi-in}\Bigg|_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi}\frac{e^{\pi^2}e^{-i\pi n} - e^{-\pi^2}e^{i\pi n}}{\pi-in} = \frac{(-1)^n\sinh(\pi^2)}{\pi(\pi-in)}.$$

So, the Fourier series is

$$\sum_{n\in\mathbb{Z}} \frac{(-1)^n \sinh(\pi^2)}{\pi(\pi - in)} e^{inx}.$$

Let us consider the Parseval equation which says

$$\sum_{n\in\mathbb{Z}} |c_n|^2 ||e^{inx}||^2 = \int_{-\pi}^{\pi} |e^{\pi x}|^2 dx.$$

On the left side when we compute

$$|c_n|^2 = \frac{(\sinh(\pi^2))^2}{\pi^2(\pi^2 + n^2)}, \quad ||e^{inx}||^2 = 2\pi.$$

On the right side we compute

$$\int_{-\pi}^{\pi} e^{2\pi x} dx = \frac{e^{2\pi^2} - e^{2\pi^2}}{2\pi} = \frac{\sinh(2\pi^2)}{\pi}.$$

So, we have the equality

$$\sum_{n\in\mathbb{Z}} \frac{(\sinh(\pi^2))^2}{\pi^2(\pi^2 + n^2)} 2\pi = \frac{\sinh(2\pi^2)}{\pi}.$$

For each $n = \pm k$ where $k \geq 1$ there are two terms in the sum on the left whose value are the same. The only loner is the term with $n = 0$. So we write out the sum on the left:

$$\frac{(\sinh(\pi^2))^2}{\pi^2(\pi^2 + 0)} 2\pi + 2\sum_{n\geq 1} \frac{(\sinh(\pi^2))^2}{\pi^2(\pi^2 + n^2)} 2\pi$$

$$= \frac{2\sinh(\pi^2)^2}{\pi^3} + \frac{4\sinh(\pi^2)^2}{\pi} \sum_{n\geq 1} \frac{1}{\pi^2 + n^2}.$$

Recalling the other side, we have the equality:

$$\frac{2\sinh(\pi^2)^2}{\pi^3} + \frac{4\sinh(\pi^2)^2}{\pi} \sum_{n\geq 1} \frac{1}{\pi^2 + n^2} = \frac{\sinh(2\pi^2)}{\pi}.$$

Solving for the sum we want to compute, first we can eliminate the π from downstairs, and also use the double angle formula for the hyperbolic sine to have the equation

$$\frac{2\sinh(\pi^2)^2}{\pi^2} + 4\sinh(\pi^2)^2 \sum_{n\geq 1} \frac{1}{\pi^2+n^2} = 2\sinh(\pi^2)\cosh(\pi^2).$$

We can divide everywhere by $2\sinh(\pi^2)$ which is certainly not zero obtaining

$$\frac{\sinh(\pi^2)}{\pi^2} + 2\sinh(\pi^2) \sum_{n\geq 1} \frac{1}{\pi^2+n^2} = \cosh(\pi^2).$$

Now we solve for the sum:

$$\frac{1}{2\sinh(\pi^2)}\left(\cosh(\pi^2) - \frac{\sinh(\pi^2)}{\pi^2}\right) = \sum_{n\geq 1} \frac{1}{\pi^2+n^2}.$$

If we are so inclined, we can tidy up the left side, to obtain

$$\frac{\coth(\pi^2)}{2} - \frac{1}{2\pi^2} = \sum_{n\geq 1} \frac{1}{\pi^2+n^2}.$$

A small reality check, observing that $\coth(\pi^2) > 1$, which guarantees that the expression on the left is positive, is reassuring.

Points:

(a) (2p) Correct definition of Fourier coefficient c_n for the function $e^{\pi x}$.

(b) (2p) Correctly computing these coefficients.

(c) (4p) Choosing *either* to use Parseval and getting that freaking right, what the theorem says, *or* choosing to use the theorem on pointwise convergence of Fourier series and getting that freaking right, what the theorem says. (This is basically all or nothing, either you know what these theorems say or you don't. No partial credit here, cause a wrongly remembered theorem is worthless).

(d) (2p) Solving for the sum and getting it right.

6. (a) Bestäm om gränsvärdet finns eller inte och förklära varför (determine whether or not the following limit exists and give a reason for your answer):

$$\lim_{n\to\infty} A_n, \quad A_n := \int_{-\pi}^{\pi} inx^2 e^{-inx} dx.$$

(5p)

Well, did you notice the choice of theory items. I hope that was not too great of a hint... You see, what we have above is 2π times the Fourier coefficient of the derivative of (x^2). The derivative of x^2 is $2x$. This is a perfectly integrable function on the interval $[-\pi, \pi]$. So, Bessel's inequality (and also remember the previous problem, where we used Parseval's equality... that was also intended as a hint)... These two facts both imply that the Fourier coefficients of $2x$ tend to zero. So, if we multiply them by 2π then they also tend to zero. Hence the limit above is zero.

Points: this is basically all or nothing. You either know that the limit exists and give a correct reason, or you don't. So 5p or 0p for this part.

(b) Beräkna:

$$\sum_{n\in\mathbb{Z}} A_n e^{42i\pi n/4}.$$

(5p)

For this problem we need to know to what does the Fourier series of the function $2x$ converge. When we create a Fourier series, we create a 2π periodic function. The Fourier series for $2x$ is

$$\sum_{n\in Z} \frac{A_n}{2\pi} e^{inx}.$$

At a point like $x = 42\pi/4$ we compute

$$\frac{42\pi}{4} = \frac{21\pi}{2} = 10\pi + \frac{\pi}{2} = 2(5\pi) + \frac{\pi}{2}.$$

Using the 2π periodicity, the Fourier series will converge to the value of $2x$ at the point $x = \frac{\pi}{2}$. So,

$$\sum_{n\in Z} \frac{A_n}{2\pi} e^{in42\pi/4} = \pi \implies \sum_{n\in\mathbb{Z}} A_n e^{42i\pi n/4} = 2\pi^2.$$

Points:

i. (2p) Recognizing that this is evaluating the Fourier series, and correctly identifying the function whose Fourier series it is (basically $2\pi(2x)$).
ii. (2p) Correctly using the 2π periodicity to figure out where to evaluate the function.
iii. (1p) Doing the algebra correctly to get the right answer in the end.

7. Låt f vara en 2π-periodisk funktion med $f \in C^1(\mathbb{R})$. Bevisa att Fourierkoefficienterna c_n av f och Fourierkoefficienterna c_n' av f' uppfyller

$$c_n' = inc_n.$$

(Assume that f is a 2π periodic smoothly differentiable function on $\mathbb{R}$. Prove that the Fourier coefficients, c_n of f and c_n' of f' satisfy $c_n' = inc_n$).

(10p)

Please see the theory proofs document!

Points:

(a) (6p) The idea to use integration by parts. Trying to differentiate the series termwise will give a big fat 0 on this problem because that argument is circular. This part is basically all or nothing (6p or 0p).
(b) (4p) Actually doing the integration by parts correctly. Each silly mistake or completely unjustified step will lose one point (until a max of 4 points can be lost from this part).

8. Låt $\{\phi_n\}_{n\in\mathbb{N}}$ vara ortonormala i ett Hilbert-rum, H. Bevisa att följande tre är ekvivalenta: (Prove that the three conditions below are equivalent statements in a Hilbert space H.)

$$(1) \quad f \in H \text{ och } \langle f, \phi_n \rangle = 0 \forall n \in \mathbb{N} \implies f = 0.$$

$$(2) \quad f \in H \implies f = \sum_{n\in\mathbb{N}} \langle f, \phi_n \rangle \phi_n.$$

$$(3) \quad ||f||^2 = \sum_{n \in \mathbb{N}} |\langle f, \phi_n \rangle|^2 .$$

(10 p)

Please see the theory proofs document!
Points:

1. (2p) The idea to prove $1 \implies 2 \implies 3 \implies 1$.
2. (3p) Proving 1 implies 2.
3. (3p) Proving 2 implies 3.
4. (2p) Proving 3 implies 1.

So now you can check for yourself to verify that these rules of grading were precisely followed on each exercise. It is rare, but possible, that a mistake could occur, so if you find anything which is inconsistent with this point scheme, please let us know and we shall correct it! ♡

Fourier transforms
In these formulas below $a > 0$ and $c \in \mathbb{R}$.

$f(x)$	$\hat{f}(\xi)$
$f(x-c)$	$e^{-ic\xi}\hat{f}(\xi)$
$e^{ixc}f(x)$	$\hat{f}(\xi - c)$
$f(ax)$	$a^{-1}\hat{f}(a^{-1}\xi)$
$f'(x)$	$i\xi\hat{f}(\xi)$
$xf(x)$	$i(\hat{f})'(\xi)$
$(f * g)(x)$	$\hat{f}(\xi)\hat{g}(\xi)$
$f(x)g(x)$	$(2\pi)^{-1}(\hat{f} * \hat{g})(\xi)$
$e^{-ax^2/2}$	$\sqrt{2\pi/a}e^{-\xi^2/(2a)}$
$(x^2+a^2)^{-1}$	$(\pi/a)e^{-a\|\xi\|}$
$e^{-a\|x\|}$	$2a(\xi^2+a^2)^{-1}$
$\chi_a(x) = \begin{cases} 1 & \|x\| < a \\ 0 & \|x\| > a \end{cases}$	$2\xi^{-1}\sin(a\xi)$
$x^{-1}\sin(ax)$	$\pi\chi_a(\xi) = \begin{cases} \pi & \|\xi\| < a \\ 0 & \|\xi\| > a \end{cases}$

$$H(t) := \begin{cases} 0 & t < 0 \\ 1 & t > 0 \end{cases}$$

Laplace transforms

In these formulas below, $a > 0$ and $c \in \mathbb{C}$.

$H(t)f(t)$	$\widetilde{f}(z)$
$H(t-a)f(t-a)$	$e^{-az}\widetilde{f}(z)$
$H(t)e^{ct}f(t)$	$\widetilde{f}(z-c)$
$H(t)f(at)$	$a^{-1}\widetilde{f}(a^{-1}z)$
$H(t)f'(t)$	$z\widetilde{f}(z) - f(0)$
$H(t)\int_0^t f(s)ds$	$z^{-1}\widetilde{f}(z)$
$H(t)(f * g)(t)$	$\widetilde{f}(z)\widetilde{g}(z)$
$H(t)t^{-1/2}e^{-a^2/(4t)}$	$\sqrt{\pi/z}e^{-a\sqrt{z}}$
$H(t)t^{-3/2}e^{-a^2/(4t)}$	$2a^{-1}\sqrt{\pi}e^{-a\sqrt{z}}$
$H(t)J_0(\sqrt{t})$	$z^{-1}e^{-1/(4z)}$
$H(t)\sin(ct)$	$c/(z^2+c^2)$
$H(t)\cos(ct)$	$z/(z^2+c^2)$
$H(t)e^{-a^2t^2}$	$(\sqrt{\pi}/(2a))e^{z^2/(4a^2)}\,\mathrm{erfc}(z/(2a))$
$H(t)\sin(\sqrt{at})$	$\sqrt{\pi a/(4z^3)}e^{-a/(4z)}$

Lycka till! May the force be with you! ♡ Julie Rowlett.