Fourieranalys MVE030 och Fourier Metoder MVE290 27.augusti.2019
Betygsgränser: 3: 40 poäng, 4: 53 poäng, 5: 67 poäng.
Maximalt antal poäng: 80.
Hjälpmedel: BETA.
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1. Lös problemet:

$$\begin{cases} u(0,t) = 0 & t > 0 \\ u_t(x,t) - u_{xx}(x,t) = 0 & t, x > 0 \\ u(x,0) = xe^{-x} & x > 0 \end{cases}$$

Does this look familiar? Indeed it is quite similar to the first problem on the exam in March. So hopefully by now you can solve this type of problem! To figure out what we should do, let's investigate the boundary condition. The boundary condition is that

$$u(0,t) = 0.$$

To achieve such a condition, we should extend the initial data oddly. The initial data is the function

$$xe^{-x}, \quad x > 0.$$

This is in $\mathcal{L}^2$ for $x > 0$. Extending oddly, we write

$$f_o(x) = f(x) = xe^{-x}, \quad x > 0, \quad f_o(x) = -f(-x) = -xe^{x} \quad x < 0.$$

Note that this function is still in $\mathcal{L}^2$, in fact it is simply the function $|x|e^{-|x|}$.

Let's apply the Fourier transform to the PDE in the x variable:

$$\hat{u}_t(\xi,t) - \widehat{u_{xx}}(\xi,t) = 0.$$

The properties of the Fourier transform say that

$$\widehat{u_{xx}}(\xi,t) = (-i\xi)^2\hat{u}(\xi,t),$$

so our equation becomes

$$\hat{u}_t(\xi,t) + \xi^2 \hat{u}(\xi,t) = 0 \implies \hat{u}(\xi,t) = a(\xi)e^{-\xi^2 t}.$$

Above we have solved the ODE for the Fourier transform where the ODE variable is the variable t, and the variable ξ is an independent variable. The initial condition is that

$$\hat{u}(\xi,0) = a(\xi) = \hat{f}_o(\xi).$$

So,

$$\hat{u}(\xi,t) = \hat{f}_o(\xi)e^{-\xi^2 t}.$$

The Fourier transform sends a convolution to a product. We look at the table to find a function whose Fourier transform is $e^{-\xi^2 t}$. We know a function whose Fourier transform is $\hat{f}_o(\xi)$, simply f_o. So,

$$u(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} f_o(y) e^{-(x-y)^2/(4t)} dy.$$

To put this in terms of the original function, and verify the boundary condition, we recall the definition of f_o as being an odd extension, so

$$u(x,t) = \frac{1}{\sqrt{4\pi t}} \left(\int_{-\infty}^0 f_o(y) e^{-(x-y)^2/(4t} dy + \int_0^\infty f(y) e^{-(x-y)^2/(4t)} dy \right).$$

We can turn the integral on the negative real axis into an integral on the positive real axis. To do this, let $z = -y$, then

$$\int_{-\infty}^0 f_o(y) e^{-(x-y)^2/(4t} dy = \int_{\infty}^0 f_o(-z) e^{-(x+z)^2/(4t)} (-dz) = \int_0^\infty f_o(-z) e^{-(x+z)^2/(4t)} dz.$$

Since

$$f_o(-z) = -f_o(z) \quad z > 0,$$

this is

$$-\int_0^\infty f(z) e^{-(x+z)^2/(4t)} dz.$$

Now, the name of the variable of integration is irrelevant, so we may as well re-name it back to y, and then we have

$$u(x,t) = \frac{1}{\sqrt{4\pi t}} \int_0^\infty f(y) \left(e^{-(x-y)^2/(4t)} - e^{-(x+y)^2/(4t)} \right) dy$$

$$= \frac{1}{\sqrt{4\pi t}} \int_0^\infty y e^{-y} \left(e^{-(x-y)^2/(4t)} - e^{-(x+y)^2/(4t)} \right) dy$$

If $x = 0$ then the two terms in parentheses cancel. So we see that the boundary condition is satisfied. Since we worked always with $\mathcal{L}^2$ functions, the convolution approximation theorem guarantees that the initial condition is also satisfied.

Since it might be helpful, here is basically how partial credit will be dished out. In case any of these items is somewhat messed up, but half-right, you'd get 1p instead of 2p.

(a) (2p) Using Fourier transform method in the x variable.

(b) (2p) Extending the initial condition oddly.

(c) (2p) Correctly Fourier transforming the PDE.

(d) (2p) Solving the ODE for the Fourier transform of the solution correctly.

(e) (2p) Correctly inverting the Fourier transform to obtain the solution (going backwards correctly).

2. Lös problemet:

$$\begin{cases} u(0,t) = \sin(t)e^t & t > 0 \\ u_t(x,t) - u_{xx}(x,t) = 0 & t, x > 0 \\ u(x,0) = 0 & x > 0 \end{cases}$$

This should also look pretty familiar. In this case we have that strange $x = 0$ boundary condition which depends on t. This tells us to use Laplace transform methods. We Laplace transform the PDE in the t variable:

$$\widetilde{u_t}(x,z) - \widetilde{u}_{xx}(x,z) = 0.$$

We use the properties of the Laplace transform and the nice homogeneous initial condition to obtain:

$$z\widetilde{u}(x,z) - \widetilde{u}_{xx}(x,z) = 0.$$

We solve this ODE to obtain:

$$\widetilde{u}(x,z) = a(z)e^{-x\sqrt{z}} + b(z)e^{x\sqrt{z}}.$$

The properties of the Laplace transform imply (indeed it was a Theorem) that anything which is Laplace-transformable will $\to 0$ as the real part of z tends to infinity. For $x > 0$ (which it is since we work in the positive real line on this problem) the second term will not satisfy that unless b has some really great decay properties. However b doesn't depend on x so if $x \to \infty$ also, then b cannot save this term from growing exponentially. Thus, we try to solve the problem using only the other term. The boundary condition says:

$$\widetilde{u}(0,z) = \widetilde{\sin(t)e^t}(z) = a(z) \implies \widetilde{u}(x,z) = \widetilde{\sin(t)e^t}(z)e^{-x\sqrt{z}}.$$

We know that the Laplace transform takes a convolution to a product. We know where the first term came from, so we look for a function whose Laplace transform is $e^{-x\sqrt{z}}$. We look at the lovely table. We see that to get $2a^{-1}\sqrt{\pi}e^{-a\sqrt{z}}$ as the Laplace transform we should start with $H(t)t^{-3/2}e^{-a^2/(4t)}$. So with our problem, we would want $a = x$, and to obtain $e^{-x\sqrt{z}}$ as the Laplace transform we should start with

$$\frac{x}{2\sqrt{\pi}t^{3/2}}H(t)e^{-x^2/(4t)}.$$

Hence

$$u(x,t) = \int_{\mathbb{R}} H(s)\sin(s)e^s H(t-s)\frac{x}{2\sqrt{\pi}(t-s)^{3/2}}e^{-x^2/(4(t-s))}ds.$$

This is because the Laplace transform is in the t variable, so that's the variable for the convolution, and also because the Laplace transform needs the functions inside to be zero for negative values (hence the Heavyside factors). With these Heavyside factors in mind, we obtain

$$u(x,t) = \int_0^t \sin(s)e^s \frac{x}{2\sqrt{\pi}(t-s)^{3/2}}e^{-x^2/(4(t-s))}ds.$$

Here is how partial credit will be dished out. In case any of these items is somewhat messed up, but half-right, you'd get 1p instead of 2p. The main idea is that you get points for things you do which bring you closer to the solution. If you do something which will not get you any closer to the solution, no points are awarded for such useless steps.

(a) (2p) Choosing to use Laplace transform methods (in the correct variable of course).

(b) (2p) Correctly Laplace transforming the PDE.

(c) (2p) Solving the ODE for the Laplace transform of the solution correctly to get the general solution.

(d) (2p) Discarding the non-Laplace-transformable part of the solution and using the BC to determine the Laplace transform of the solution to the PDE. (Basically going from the general solution of the ODE to the particular solution correctly here).

(e) (2p) Correctly inverting the Laplace transform to obtain the solution (going backwards correctly).

3. Lös ekvationen:

$$u(t) + \int_{-\infty}^{\infty} e^{-(t-\tau)^2} u(\tau) d\tau = e^{-|t|}.$$

The déjà vu continues. This is intentional. Each year three exams will be created which are all rather similar. The idea is that if you try once but do not succeed, you study for the second exam. You should study carefully the problems which foiled you on the first exam. Hopefully the second time around you will succeed with these problems. If not, keep calm and study on. Again, you should study the problems that you missed in the first and second exams. By the way, if you are having trouble with these, you should ask for help, always happy to help students who are studying and trying to learn, and sympathetic to the fact that this is not easy math! Anyhow, by the third time around, we really really hope that you've mastered the problems which stumped you on the first and second exams, so that you'll pass the third exam. So, the exams will be in this sort of similar-but-not-identical-triplets, with the goal of helping you all learn the material and eventually succeed with it.

Let's continue with the solution. The second term is a convolution, and the term on the right is one of the items on our list of Fourier transforms. So let us transform this entire equation:

$$\hat{u}(\xi) + \hat{u}(\xi)\sqrt{\pi} e^{-\xi^2/4} = \frac{2}{\xi^2+1}.$$

This is because the Fourier transform of a convolution is the product of the Fourier transforms, and the Fourier transform of $e^{-|x|}$ as well as e^{-x^2} are given in the handy table at the end. We solve this equation for $\hat{u}(\xi)$:

$$\hat{u}(\xi)\left(1+\sqrt{\pi}e^{-\xi^2/4}\right) = \frac{2}{\xi^2+1} \implies \hat{u}(\xi) = \frac{2}{(\xi^2+1)(1+\sqrt{\pi}e^{-\xi^2/4})}.$$

Finally, we use the Fourier Inversion Theorem to write

$$u(x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{ix\xi}\frac{2}{(\xi^2+1)(1+\sqrt{\pi}e^{-\xi^2/4})}d\xi.$$

Points:

(a) (2p) Choosing to use Fourier transform methods.

(b) (2p) Correctly Fourier transforming the equation.

(c) (3p) Correctly solving for the Fourier transform of u.

(d) (3p) Correctly giving the Fourier inversion formula.

4. Lös problemet:

$$\begin{cases} u_{tt}(x,t) - u_{xx}(x,t) = xe^x & 0<t,\ 0<x<1 \\ u(x,0) = 0 & x \in [0,1] \\ u_t(x,0) = h(x) \in \mathcal{C}^0[0,1] & x \in [0,1] \\ u(0,t) = 0 = u(1,t) & t > 0 \end{cases}$$

Now we are in the world of bounded intervals. The boundary conditions are fantastic. The initial conditions are fine. The only issue is that the PDE is not homogeneous. However, it is *time independent.* So we can attempt to deal with this by finding a steady state (that means time independent) solution. So we first seek a function ϕ which satisfies

$$-\phi''(x) = xe^x.$$

We would also like to preserve the beautiful boundary conditions, so we politely request that

$$\phi(0) = \phi(1) = 0.$$

To solve the ODE, note that xe^x has derivative $xe^x + e^x$. So the function $xe^x - e^x$ has derivative equal to xe^x. Repeating this idea, the function $xe^x - 2e^x$ has second derivative equal to xe^x. So a particular solution to the ODE is

$$-xe^x + 2e^x.$$

Solutions to the homogeneous version of this ODE are linear functions. So a general solution is

$$\phi(x) = -xe^x + 2e^x + ax + b,$$

for some constants a and b. To achieve the boundary condition at $x = 0$, we need $b = -2$. To achieve the boundary condition at $x = 1$ we need

$$0 = -e + 2e + a + -2 \implies a = 2 - e.$$

So we define

$$\phi(x) = -xe^x + 2e^x + (2 - e)x - 2.$$

Now, we just need to solve a nicer problem:

$$\begin{cases} v_{tt}(x,t) - v_{xx}(x,t) = 0 & 0 < t,\ 0 < x < 1 \\ v(x,0) = -\phi(x) & x \in [0,1] \\ v_t(x,0) = h(x) \in \mathcal{C}^0[0,1] & x \in [0,1] \\ v(0,t) = 0 = v(1,t) & t > 0 \end{cases}.$$

Then, the full solution will be

$$u(x,t) = \phi(x) + v(x,t).$$

Note that our initial data is still beautiful, continuous, and certainly therefore in $\mathcal{L}^2(0,1)$. Moreover, the boundary conditions are fantastic (self adjoint in particular). So Fourier series methods ought to work here.

We approach the problem at hand now by separating variables writing

$$v = X(x)T(t).$$

We put this into the PDE:

$$T''(t)X(x) - X''(x)T(t) = 0.$$

We tidy it up so that all time dependent terms are on one side, and all space dependent terms are on the other side. So, to achieve this we first divide by XT and then re-arrange:

$$\frac{T''}{T} = \frac{X''}{X}.$$

Since the two sides depend on different variables, they must both be constant. So, we look for solutions to

$$\frac{T''}{T} = \text{ constant } = \frac{X''}{X}.$$

We start with the X side because its conditions are homogeneous and simple. In particular, we seek to solve

$$X'' = \lambda X, \quad X(0) = X(1) = 0.$$

If you recognize the solutions will be sines, you can "skip to the good bit." Otherwise one needs to check all cases. First case, $\lambda = 0$. Then X would be a linear function. Linear functions cannot go up and then down. They either go up, down, or lie flat. In this case, to have $X(0) = X(1) = 0$, we need the flatline zero linear function. That will not contribute anything non-zero to our solution.

In the next case $\lambda > 0$. So, the solution to the equation could be written as either a linear combination of $e^{\pm\sqrt{\lambda}x}$ or as a linear combination of hyperbolic sine and cosine. Let us use the latter, because 0 is in our interval. Writing

$$a\cosh(\sqrt{\lambda}x) + b\sinh(\sqrt{\lambda}x)$$

the condition to vanish at $x = 0$ requires that $a = 0$. The condition to vanish at $x = 1$ would require (if we want $b \neq 0$) that $\sinh(\sqrt{\lambda}) = 0$. The only real number at which the sinh vanishes is at zero. So we would need $\lambda = 0$. However that contradicts the case we are in. Therefore the case $\lambda > 0$ yields no non-zero solutions.

Finally, we have the case $\lambda < 0$. In this case the solutions are linear combinations of $\sin(\sqrt{|\lambda|}x)$ and $\cos(\sqrt{|\lambda|}x)$. The condition to vanish at zero means that there cannot be a cosine term. Moreover, the condition to vanish at $x = 1$ means that we need $\sqrt{|\lambda|}$ to be an integer multiple of π. Consequently, all solutions we find in this way are, up to constant factors,

$$X_n(x) = \sin(n\pi x), \quad \lambda_n = -n^2\pi^2.$$

This informs us what the T function must be since

$$\frac{T_n''}{T_n} = \lambda_n = -n^2\pi^2 \implies T_n(t) = \text{ a linear combination of } \sin(n\pi t) \text{ and } \cos(n\pi t).$$

In the last step, we put together all the $X_n T_n$ pairs, by the superposition principle, because the PDE is homogeneous, thereby creating our super solution:

$$v(x,t) = \sum_{n\geq 1} X_n(x) \left(a_n \cos(n\pi t) + b_n \sin(n\pi t)\right).$$

We shall need the constant factors now to guarantee that the initial conditions are satisfied. First we have the condition at $t = 0$ for the function,

$$v(x,0) = \sum_{n\geq 1} a_n X_n(x) = -\phi(x) \implies a_n = \frac{\int_0^1 -\phi(x)\overline{X_n(x)}}{\int_0^1 |X_n|^2}.$$

The reason we can expand the function $-\phi(x)$ in a Fourier X_n series is that the SLP theory guarantees that the functions X_n form an orthogonal basis for $\mathcal{L}^2$ on the interval $[0,1]$.

Next we have the condition for the derivative at zero, so

$$v_t(x,0) = \sum_{n\geq 1} b_n (n\pi) X_n(x) = h(x) \implies b_n = \frac{\int_0^1 h\overline{X}_n}{n\pi \int_0^1 |X_n|^2}.$$

Similar considerations justify the expansion of h in a Fourier X_n series. We have therefore specified all quantities in our solution.

Points:

(a) (1p) Choosing to find a steady state solution to deal with the inhomogeneity in the PDE.

(b) (2p) Correctly solving for the steady state solution to solve the inhomogeneous PDE and not screw up the nice BC.

(c) (1p) Setting up the next problem to solve correctly. (homog. PDE, modified IC, same BC, then observe full solution will be sum of these two).

(d) (2p) Choosing to use separation of variables.

(e) (2p) Obtaining the X_n part of the solution correctly.

(f) (2p) Obtaining the T_n part of the solution, in particular getting the a_n and the b_n coefficients correctly.

5. Beräkna:

$$\lim_{N\to\infty} \sum_{n=-N}^{N} \frac{1}{\pi + n}.$$

(Tips: beräkna Fourier-serien av den 2π periodiska funktionen som är lika med $\cos(\pi x)$ i intervallet $(-\pi, \pi)$.)

Let's follow the hint. The Fourier coefficients are

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos(\pi x) e^{-inx} dx.$$

For me, it is easier to turn the cosine into

$$\cos(\pi x) = \frac{e^{i\pi x} + e^{-i\pi x}}{2}.$$

So I will do this and then compute

$$c_n = \frac{1}{4\pi} \int_{-\pi}^{\pi} e^{i\pi x - inx} + e^{-i\pi x - inx} dx$$

$$= \frac{1}{4\pi(i\pi - in)} \left(e^{\pi(i\pi - in)} - e^{-\pi(i\pi - in)}\right) + \frac{1}{4\pi(-i\pi - in)} \left(e^{\pi(-i\pi - in)} - e^{-\pi(-i\pi - in)}\right).$$

$$= \frac{1}{2\pi(\pi - n)} (-1)^n \sin(\pi^2) + \frac{1}{2\pi(\pi + n)} (-1)^n \sin(\pi^2)$$

$$= \frac{(-1)^n \sin(\pi^2)}{2\pi} \left(\frac{\pi + n + (\pi - n)}{(\pi - n)(\pi + n)}\right)$$

$$= \frac{(-1)^n \sin(\pi^2)}{2\pi} \frac{2\pi}{\pi^2 - n^2}.$$

So the Fourier series for the function which is equal to $\cos(\pi x)$ in the interval $(-\pi, \pi)$ and is 2π periodic is

$$\sum_{n\in\mathbb{Z}} \frac{(-1)^n \sin(\pi^2)}{(\pi^2 - n^2)} e^{inx}.$$

Now we are going to have to think a bit. We want to use this somehow to compute the rather mysterious sum

$$\sum_{n=-N}^{N} \frac{1}{\pi + n},$$

when $N \to \infty$. Note that we can pair up terms of the form $\pm n$ for all non-zero n. When we do this we get:

$$\frac{1}{\pi} + \sum_{n=1}^{N} \frac{1}{\pi + n} + \frac{1}{\pi - n} = \frac{1}{\pi} + \sum_{n=1}^{N} \frac{\pi - n + \pi + n}{\pi^2 - n^2} = \frac{1}{\pi} + \sum_{n=1}^{N} \frac{2\pi}{\pi^2 - n^2}.$$

Hope springs eternal! To get our Fourier series looking like this we want to get rid of the pesky alternation $(-1)^n$. To do that we choose to evaluate the series at $x = \pi$. What is the limit of the series? We must use the theorem on the pointwise convergence of Fourier series. When we do this, we get that the series at $x = \pi$ converges to the sum of the left and right limits of our function. It is 2π periodic. So

$$\lim_{x\to\pi, x<\pi} \text{ is } \cos(\pi^2).$$

On the other hand

$$\lim_{x\to\pi, x>\pi} \text{ is } \lim_{x\to-\pi, x>-\pi} = \cos(-\pi^2).$$

(maybe draw a picture to see this? It is because of the 2π periodicity.) So the average of these limits gives us that the Fourier series converges to

$$\frac{\cos(\pi^2) + \cos(-\pi^2)}{2} = \sum_{n\in\mathbb{Z}} \frac{(-1)^n \sin(\pi^2)}{(\pi^2 - n^2)} e^{in\pi}.$$

Note that $e^{in\pi} = (-1)^n$. So this series simplifies to

$$\sum_{n\in\mathbb{Z}} \frac{\sin(\pi^2)}{(\pi^2 - n^2)}.$$

The terms are the same for $n = \pm 1, \pm 2, \ldots$, so the series simplifies to

$$\frac{\sin(\pi^2)}{\pi^2} + \sum_{n\geq 1} \frac{2\sin(\pi^2)}{(\pi^2 - n^2)}.$$

Hence we have the equality

$$\frac{\cos(\pi^2) + \cos(-\pi^2)}{2} = \frac{\sin(\pi^2)}{\pi^2} + \sum_{n\geq 1} \frac{2\sin(\pi^2)}{(\pi^2 - n^2)}.$$

The cosine is even, so we can simplify and re-arrange things a bit to obtain

$$\cos(\pi^2) - \frac{\sin(\pi^2)}{\pi^2} = \sin(\pi^2) \sum_{n\geq 1} \frac{2}{\pi^2 - n^2}.$$

We therefore obtain:

$$\sum_{n\geq 1} \frac{2}{\pi^2 - n^2} = \left(\cos(\pi^2) - \frac{\sin(\pi^2)}{\pi^2}\right) \frac{1}{\sin(\pi^2)} = \cot(\pi^2) - \frac{1}{\pi^2}.$$

So, we therefore have that

$$\lim_{N\to\infty} \sum_{n=-N}^{N} \frac{1}{\pi + n} = \lim_{N\to\infty} \frac{1}{\pi} + \sum_{n=1}^{N} \frac{2\pi}{\pi^2 - n^2} = \frac{1}{\pi} + \pi \left(\cot(\pi^2) - \frac{1}{\pi^2}\right) = \pi\cot(\pi^2).$$

Points:

(a) (1p) Correct definition of Fourier coefficient c_n for the function $\cos(\pi x)$.

(b) (1p) Correctly computing these coefficients.

(c) (4p) Correctly applying the theorem on pointwise convergence of Fourier series to evaluate the series at $x = \pi$.

(d) (2p) Correctly manoeuvring the series in question from the statement of the problem to make it look like the Fourier series.

(e) (2p) Solving for the sum and getting it right.

6. Visa att om $n \in \mathbb{N}$

$$\pi J_n(z) = \int_0^\pi \cos(z \sin\theta - n\theta) d\theta.$$

(Tips: nästa uppgift med $z = e^{i\theta}$.)

You know how in WatchMojo there is the "one per list?" Well this is the "one per exam" problem that is a little unpredictable. Each exam should have a problem which is a little bit original, to keep things interesting. So that you all have good chances of passing the exam, the other problems should be more on the predictable side.

We follow the hint. We use the following exercise and make the special choice of $z = e^{i\theta}$. Then you have the equality:

$$\sum_{n=-\infty}^{\infty} J_n(x) e^{in\theta} = e^{\frac{x}{2}(e^{i\theta} - \frac{1}{e^{i\theta}})} = e^{\frac{x}{2}(e^{i\theta} - e^{-i\theta})} = e^{ix\sin(\theta)}.$$

This means that the Fourier coefficients of the function which is equal to $e^{ix\sin(\theta)}$ on $(-\pi, \pi)$ and extended to be 2π periodic is equal to $J_n(x)$. So, by definition of Fourier coefficient we have

$$J_n(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ix\sin(\theta)} e^{-in\theta} d\theta.$$

Now we use Euler's formula to re-write this in terms of cosine and sine to get closer to the integral expression given in the problem

$$J_n(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos(x\sin\theta - n\theta) + i\sin(x\sin\theta - n\theta) d\theta.$$

Due to the oddness of the sine, the integral with the sine vanishes. Due to the evenness of the cosine, that term is

$$\frac{1}{\pi} \int_0^\pi \cos(x\sin\theta - n\theta) d\theta.$$

Hence in conclusion

$$\pi J_n(x) = \int_0^\pi \cos(x\sin\theta - n\theta) d\theta.$$

Points:

(a) (4p) Recognizing that when you follow the hint and set $z = e^{i\theta}$ you obtain a Fourier series.

(b) (2p) Correctly giving the definition of the Fourier series for the function $e^{ix\sin\theta}$.

(c) (2p) Using Euler's formula correctly to obtain the integral in terms of sines and cosines.

(d) (2p) Correctly simplifying the integral to obtain the expression.

7. Bevisa att för $z \neq 0$, de Bessel funktionerna uppfyller:

$$\sum_{n=-\infty}^{\infty} J_n(x) z^n = e^{\frac{x}{2}(z-\frac{1}{z})}.$$

The proof is contained in the proofs of theory items documents for the generating function for the Bessel functions.

Points:

(a) (2p) Idea to Taylor expand the exponential function on the right side.

(b) (2p) Idea to expand *each* of the two functions,

$$e^{xz/2} = \sum_{j \geq 0} \frac{\left(\frac{xz}{2}\right)^j}{j!},$$

and

$$e^{-x/(2z)} = \sum_{k \geq 0} \frac{\left(\frac{-x}{2z}\right)^k}{k!}$$

as their own Taylor series.

(c) (2p) One point each for getting these expansions right.

(d) (2p) Variable change to get a sum from $-\infty$ to ∞ rather than 0 to ∞. (1p for the idea and 1p for doing it right).

(e) (2p) Correct algebraic manipulations to make the series expansion for the Bessel functions to appear.

8. Låt $\{\phi_n\}_{n\in\mathbb{N}}$ vara en ortonormal mängd i ett Hilbert-rum, H. Om $f \in H$,

$$||f - \sum_{n\in\mathbb{N}} \langle f, \phi_n\rangle \phi_n|| \leq ||f - \sum_{n\in\mathbb{N}} c_n \phi_n||, \quad \forall \{c_n\}_{n\in\mathbb{N}} \in \ell^2,$$

och $=$ gäller $\iff$ $c_n = \langle f, \phi_n\rangle$ gäller $\forall n \in \mathbb{N}$.

The proof is contained in the proofs of theory items documents for the best approximation theorem.

(a) (2p) Defining

$$g = \sum \hat{f}_n \phi_n$$

and giving some explanation about it being an element of H. (Of course I do not care if you call it g or Bob or something else).

(b) (2p) Defining

$$\varphi = \sum c_n \phi_n$$

and giving some explanation about it being an element of H. (Of course I do not care if you call it φ or Anne or something else).

(c) (2p) The trick:

$$||f - \varphi||^2 = ||f - g + g - \varphi||^2.$$

(d) (2p) The calculation of:

$$||f - g + g - \varphi||^2 = \langle f - g + g - \varphi, f - g + g - \varphi\rangle$$

and obtaining that it simplifies to

$$||f - \varphi||^2 = ||f - g||^2 + ||g - \varphi||^2.$$

(e) (1p) Using this to conclude that

$$||f - \varphi||^2 \geq ||f - g||^2$$

with equality if and only if $||g - \varphi||^2 = 0$.

(f) (1p) Explaining that $||g - \varphi||^2 = 0$ if and only if $c_n = \hat{f}_n$ for all n.

Fourier transforms

In these formulas below $a > 0$ and $c \in \mathbb{R}$.

$f(x)$	$\hat{f}(\xi)$
$f(x-c)$	$e^{-ic\xi}\hat{f}(\xi)$
$e^{ixc}f(x)$	$\hat{f}(\xi-c)$
$f(ax)$	$a^{-1}\hat{f}(a^{-1}\xi)$
$f'(x)$	$i\xi\hat{f}(\xi)$
$xf(x)$	$i(\hat{f})'(\xi)$
$(f*g)(x)$	$\hat{f}(\xi)\hat{g}(\xi)$
$f(x)g(x)$	$(2\pi)^{-1}(\hat{f}*\hat{g})(\xi)$
$e^{-ax^2/2}$	$\sqrt{2\pi/a}e^{-\xi^2/(2a)}$
$(x^2+a^2)^{-1}$	$(\pi/a)e^{-a\lvert\xi\rvert}$
$e^{-a\lvert x\rvert}$	$2a(\xi^2+a^2)^{-1}$
$\chi_a(x) = \begin{cases} 1 & \lvert x\rvert < a \\ 0 & \lvert x\rvert > a \end{cases}$	$2\xi^{-1}\sin(a\xi)$
$x^{-1}\sin(ax)$	$\pi\chi_a(\xi) = \begin{cases} \pi & \lvert\xi\rvert < a \\ 0 & \lvert\xi\rvert > a \end{cases}$

$$H(t) := \begin{cases} 0 & t < 0 \\ 1 & t > 0 \end{cases}$$

Laplace transforms

In these formulas below, $a > 0$ and $c \in \mathbb{C}$.

$H(t)f(t)$	$\widetilde{f}(z)$
$H(t-a)f(t-a)$	$e^{-az}\widetilde{f}(z)$
$H(t)e^{ct}f(t)$	$\widetilde{f}(z-c)$
$H(t)f(at)$	$a^{-1}\widetilde{f}(a^{-1}z)$
$H(t)f'(t)$	$z\widetilde{f}(z) - f(0)$
$H(t)\int_0^t f(s)ds$	$z^{-1}\widetilde{f}(z)$
$H(t)(f*g)(t)$	$\widetilde{f}(z)\widetilde{g}(z)$
$H(t)t^{-1/2}e^{-a^2/(4t)}$	$\sqrt{\pi/z}e^{-a\sqrt{z}}$
$H(t)t^{-3/2}e^{-a^2/(4t)}$	$2a^{-1}\sqrt{\pi}e^{-a\sqrt{z}}$
$H(t)J_0(\sqrt{t})$	$z^{-1}e^{-1/(4z)}$
$H(t)\sin(ct)$	$c/(z^2+c^2)$
$H(t)\cos(ct)$	$z/(z^2+c^2)$
$H(t)e^{-a^2t^2}$	$(\sqrt{\pi}/(2a))e^{z^2/(4a^2)}\operatorname{erfc}(z/(2a))$
$H(t)\sin(\sqrt{at})$	$\sqrt{\pi a/(4z^3)}e^{-a/(4z)}$

Lycka till! May the force be with you! ♡ Julie Rowlett.