Recall :

Let $\Omega \subseteq \mathbb{R}^2$ be a path-connected domain. A C^1-vector field $\mathbb{F} : \Omega \to \mathbb{R}^2$, $\mathbb{F} = (P, Q)$, is said to be

(i) irrotational, if $\dfrac{\partial Q}{\partial x} = \dfrac{\partial P}{\partial y}$

(ii) conservative, if $\exists\ \phi : \Omega \to \mathbb{R}$ such that $\mathbb{F} = \nabla \phi$.

Let $X = X_{\Omega} = \{ \mathbb{F} : \Omega \to \mathbb{R}^2 : \mathbb{F} \text{ is } C^1 \}$

Note that X is a vector space (we can add & scalar-multiply vector-valued functions pointwise in each component)

Let $V = \{ \mathbb{F} \in X : \mathbb{F} \text{ is irrotational} \}$
$W = \{ \mathbb{F} \in X : \mathbb{F} \text{ is conservative} \}$

From the definitions (i), (ii) it is easy to see that V and W are subspaces of X (ie. both are closed under addition & scalar multiplication)

Theorem 9.4.4 says that $W \subseteq V$ for any region Ω and Theorem 9.4.5 says that $W = V$ if Ω is simply connected

I will now outline a proof that

$$\dim(V/W) = 1 \quad \text{if } \Omega = \mathbb{R}^2 \setminus \{(0,0)\}.$$

Step 1 : let $\mathbb{F} \in V$ and let γ be any simple, closed curve in Ω s.t. $(0,0)$ is on the outside of γ (recall that Jordan's Theorem implies that "inside" and "outside" of γ are well-defined). Then $\oint_\gamma \mathbb{F} \cdot d\mathbb{r} = 0$.

Proof Since $(0,0)$ is outside γ, the inside of γ lies entirely within Ω. So we can ~~apply~~ Green's sats:

$$\oint_\gamma \mathbb{F} \cdot d\mathbb{r} = \iint_{\text{inside}(\gamma)} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy$$

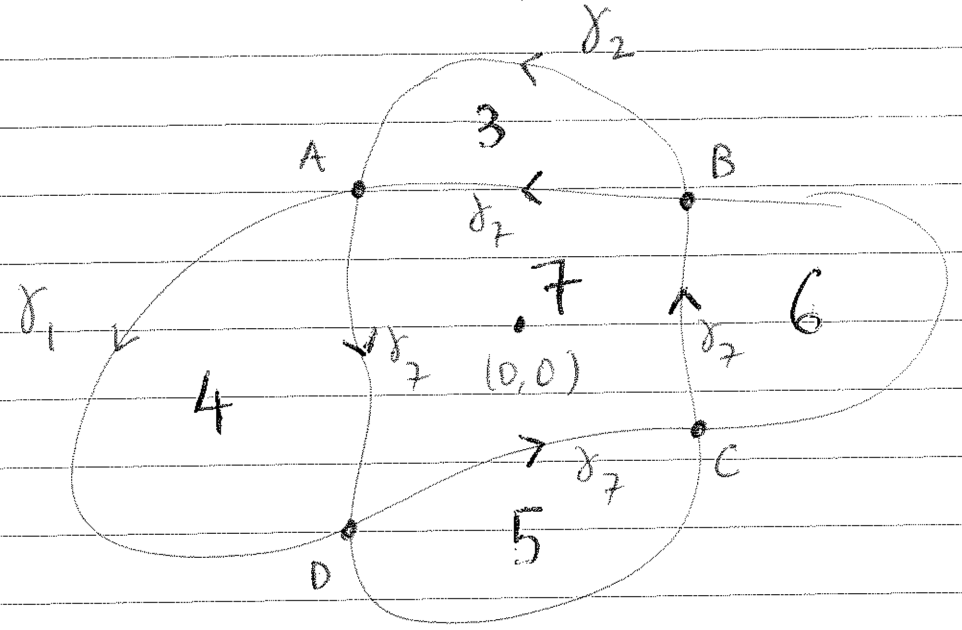
$$= 0, \quad \text{since } \mathbb{F} \in V \qquad \boxtimes$$

Step 2 Let $\mathbb{F} \in V$ and let γ_1, γ_2 be simple, closed curves in Ω such that $(0,0)$ lies inside both. Then

$$\oint_{\gamma_1} \mathbb{F} \cdot d\mathbb{r} = \oint_{\gamma_2} \mathbb{F} \cdot d\mathbb{r},$$

provided γ_1, γ_2 are each positively oriented

"Proof": Consider the following picture:



γ_1 is the "long sideways" curve passing anti-clockwise through B, A, D, C, in that order.

γ_2 is the "long vertical" curve passing through B, A, D, C in that order.

Now define $\gamma_3, \gamma_4, \gamma_5, \gamma_6, \gamma_7$ as follows:

γ_3: the pos. or. curve bounding region #3, thus passing through A & B

γ_4: the pos. or. curve bounding region #4, thus passing through A & D

γ_5: the pos. or. curve bounding region #5, thus passing through D & C

γ_6 : the pos. or. curve bounding region #6, thus passing through B & C.

γ_7 : the pos. or. curve bounding the interior region #7, thus passing through B, A, D, C in that order.

Observe that:

$$\gamma_1 = \gamma_6 + \gamma_7 + \gamma_4$$

$$\gamma_2 = \gamma_3 + \gamma_7 + \gamma_5$$

Hence:

$$\oint_{\gamma_1} \mathbb{F} \cdot d\mathbb{r} = \oint_{\gamma_6} \mathbb{F} \cdot d\mathbb{r} + \oint_{\gamma_7} \mathbb{F} \cdot d\mathbb{r} + \oint_{\gamma_4} \mathbb{F} \cdot d\mathbb{r}$$

and

$$\oint_{\gamma_2} \mathbb{F} \cdot d\mathbb{r} = \oint_{\gamma_3} \mathbb{F} \cdot d\mathbb{r} + \oint_{\gamma_7} \mathbb{F} \cdot d\mathbb{r} + \oint_{\gamma_5} \mathbb{F} \cdot d\mathbb{r}$$

However, $(0,0)$ lies on the outside of each of γ_3, γ_4, γ_5 and γ_6 so, by Step 1, the integral of $\mathbb{F}$ along each of these curves is zero. Hence,

$$\oint_{\gamma_1} \mathbb{F} \cdot d\mathbb{r} = \oint_{\gamma_2} \mathbb{F} \cdot d\mathbb{r} = \oint_{\gamma_7} \mathbb{F} \cdot d\mathbb{r} \quad \boxtimes$$

Step 3 : let $\phi : V \to \mathbb{R}$ be the map defined by

$$\phi(\mathbb{F}) = \oint_\gamma \mathbb{F} \cdot d\mathbb{r},$$

where γ is any simple, closed, positively oriented curve for which $(0,0)$ lies on the inside of γ.

By Step 2, the map ϕ is well-defined, ie: the line integral is independent of the choice of γ.

It is obvious that ϕ is a linear map between real vector spaces, ie: $\phi(\alpha \cdot \mathbb{F} + \beta \cdot \mathbb{G}) = \alpha\, \phi(\mathbb{F}) + \beta\, \phi(\mathbb{G})$.

Let $W^* = \ker(\phi)$. Since ϕ is linear, $V/W^* \simeq \operatorname{Im}(\phi)$. But $\operatorname{Im}(\phi) \subseteq \mathbb{R}$, a one-dimensional space. So $\dim(V/W^*) = 1$ if and only if there exists some $\mathbb{F} \in V$ and some γ s.t. $\oint_\gamma \mathbb{F} \cdot d\mathbb{r} \neq 0$. But we

saw an example of such a vector field in the lectures, ie:

$$\mathbb{F} = \mathbb{B} = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right)$$

Thus $\dim(V/W^*) = 1$.

But what is W^*? It consists of those irrotational $\mathbb{F}$ s.t. $\oint_\gamma \mathbb{F} \cdot d\mathbb{r} = 0$,

even when γ wraps around $(0,0)$. In other words, $W^* = W$, by Theorem 9.4.3.

$\boxed{\checkmark}$

<u>Remark</u>: The idea of the proof above can be generalised to show that, for any region $\Omega \subseteq \mathbb{R}^2$,

$$\dim(V/W) = \text{"number of holes in } \gamma\text{"}$$

The analog of the map ϕ above would be a map

$$\phi : \mathbb{F} \mapsto \left(\oint_{\gamma_1} \mathbb{F} \cdot d\mathbb{r}, \ldots, \oint_{\gamma_k} \mathbb{F} \cdot d\mathbb{r} \right) \in \mathbb{R}^k,$$

where k is the "number of holes" and γ_i is a simple, closed positively oriented curve wrapping around the i^{th} hole, but not around any other hole.