10.5 Quadric Surfaces

The most general second-degree equation in three variables is

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Iz = J.$$

We will not attempt the (rather difficult) task of classifying all the surfaces that can be represented by such an equation, but will examine some interesting special cases. Let us observe at the outset that if the above equation can be factored in the form

$$(A_1x + B_1y + C_1z - D_1)(A_2x + B_2y + C_2z - D_2) = 0,$$

then the graph is, in fact, a pair of planes,

$$A_1x + B_1y + C_1z = D_1 \qquad \text{and} \qquad A_2x + B_2y + C_2z = D_2,$$

or one plane if the two linear equations represent the same plane. This is considered a degenerate case. Where such factorization is not possible, the surface (called a **quadric surface**) will not be flat, although there may still be straight lines that lie on the surface. Nondegenerate quadric surfaces fall into the following six categories.

Spheres. The equation $x^2 + y^2 + z^2 = a^2$ represents a sphere of radius a centred at the origin. More generally,

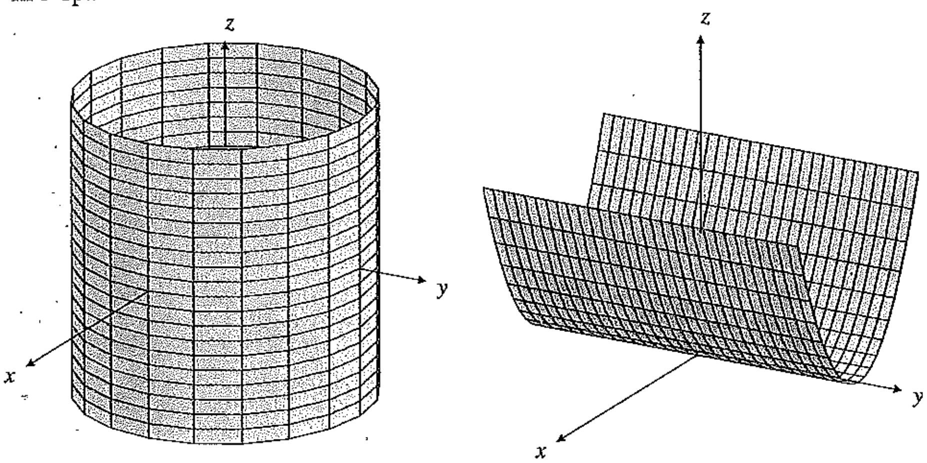
$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = a^2$$

represents a sphere of radius a centred at the point (x_0, y_0, z_0). If a quadratic equation in x, y, and z has equal coefficients for the x^2, y^2, and z^2 terms and has no other second-degree terms, then it will represent, if any surface at all, a sphere. The centre can be found by completing the squares as for circles in the plane.

Cylinders. The equation $x^2 + y^2 = a^2$, being independent of z, represents a **right-circular cylinder** of radius a and axis along the z-axis. (See Figure 10.34(a).) The intersection of the cylinder with the horizontal plane $z = k$ is the circle with equations

$$\begin{cases} x^2 + y^2 = a^2 \\ z = k. \end{cases}$$

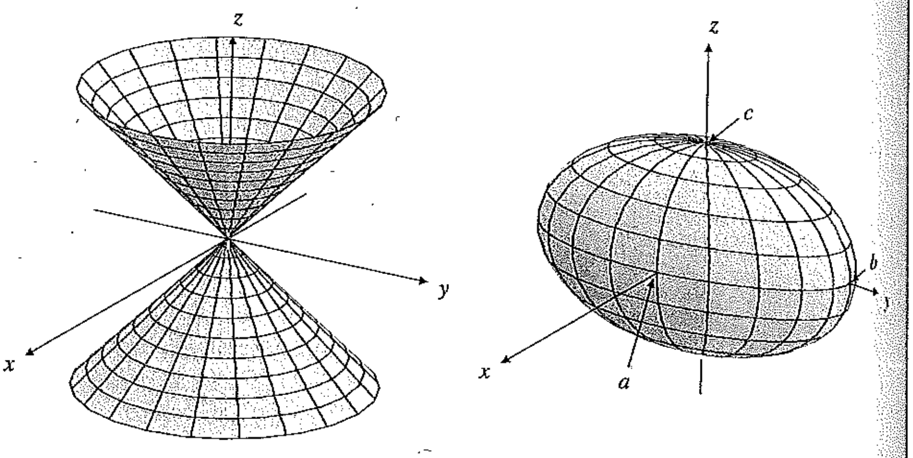
Quadric cylinders also come in other shapes: elliptic, parabolic, and hyperbolic. For instance, $z = x^2$ represents a parabolic cylinder with vertex line along the y-axis. (See Figure 10.34(b).) In general, an equation in two variables only will represent a cylinder in 3-space.



(a) (b)

Figure 10.34
(a) The circular cylinder $x^2 + y^2 = a^2$
(b) The parabolic cylinder $z = x^2$

Cones. The equation $z^2 = x^2 + y^2$ represents a **right-circular cone** with axis along the z-axis. The surface is generated by rotating about the z-axis the line $z = y$ in the yz-plane. This *generator* makes an angle of 45° with the axis of the cone. Cross-sections of the cone in planes parallel to the xy-plane are circles. (See Figure 10.35(a).) The equation $x^2 + y^2 = a^2z^2$ also represents a right-circular cone with vertex at the origin and axis along the z-axis but having semi-vertical angle $\alpha = \tan^{-1} a$. A circular cone has plane cross-sections that are elliptical, parabolic, and hyperbolic. Conversely, any nondegenerate quadric cone has a direction perpendicular to which the cross-sections of the cone are circular. In that sense, every quadric cone is a circular cone, although it may be *oblique* rather than right-circular in that the line joining the centres of the circular cross-sections need not be perpendicular to those cross-sections. (See Exercise 24.)



(a) (b)

Figure 10.35

(a) The circular cone $a^2z^2 = x^2 + y^2$

(b) The ellipsoid $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$

Ellipsoids. The equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

represents an **ellipsoid** with *semi-axes* a, b, and c. (See Figure 10.35(b).) The surface is oval, and it is enclosed inside the rectangular parallelepiped $-a \le x \le a$, $-b \le y \le b$, $-c \le z \le c$. If $a = b = c$, the ellipsoid is a sphere. In general, all plane cross-sections of ellipsoids are ellipses. This is easy to see for cross-sections parallel to coordinate planes, but somewhat harder to see for other planes.

Paraboloids. The equations

$$z = \frac{x^2}{a^2} + \frac{y^2}{b^2} \qquad \text{and} \qquad z = \frac{x^2}{a^2} - \frac{y^2}{b^2}$$

represent, respectively, an **elliptic paraboloid** and a **hyperbolic paraboloid**. (See Figure 10.36(a) and (b).) Cross-sections in planes $z = k$ (k being a positive constant) are ellipses (circles if $a = b$) and hyperbolas, respectively. Parabolic reflective mirrors have the shape of circular paraboloids. The hyperbolic paraboloid is a **ruled surface**. (A ruled surface is one through every point of which there passes a straight line lying wholly on the surface. Cones and cylinders are also examples of ruled surfaces.) There are two one-parameter families of straight lines that lie on the hyperbolic paraboloid:

$$\begin{cases} \lambda z = \dfrac{x}{a} - \dfrac{y}{b} \\ \dfrac{1}{\lambda} = \dfrac{x}{a} + \dfrac{y}{b} \end{cases} \qquad \text{and} \qquad \begin{cases} \mu z = \dfrac{x}{a} + \dfrac{y}{b} \\ \dfrac{1}{\mu} = \dfrac{x}{a} - \dfrac{y}{b} \end{cases},$$

where λ and μ are real parameters. Every point on the hyperbolic paraboloid lies on one line of each family.

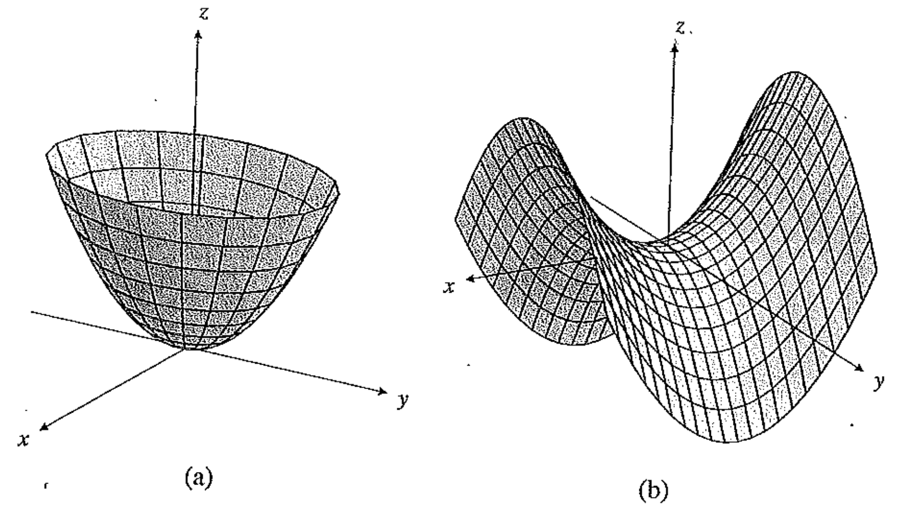


Figure 10.36

(a) The elliptic paraboloid $z = \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2}$

(b) The hyperbolic paraboloid $z = \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2}$

Hyperboloids. The equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

represents a surface called a **hyperboloid of one sheet**. (See Figure 10.37(a).) The equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1$$

represents a **hyperboloid of two sheets.** (See Figure 10.37(b).) Both surfaces

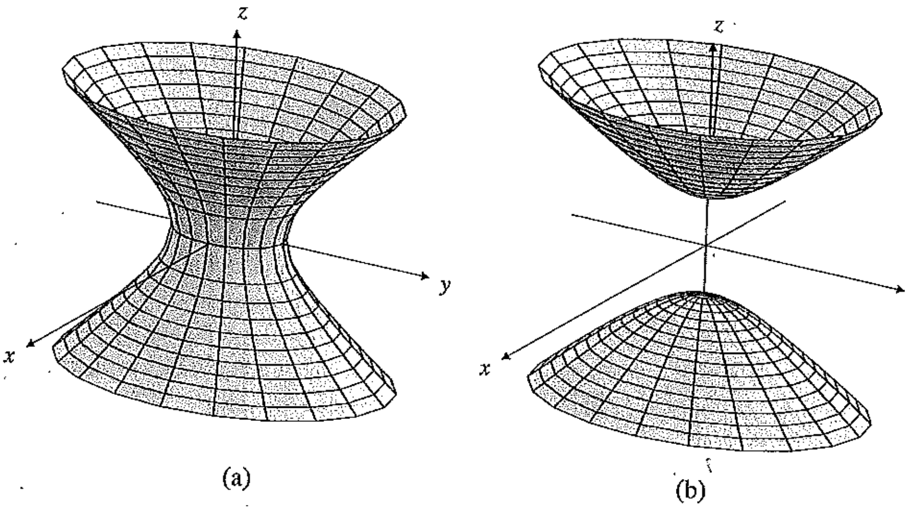


Figure 10.37

(a) The hyperboloid of one sheet $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 1$

(b) The hyperboloid of two sheets $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = -1$

have elliptical cross-sections in horizontal planes and hyperbolic cross-sections in vertical planes. Both are *asymptotic* to the elliptic cone with equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2};$$

they approach arbitrarily close to the cone as they recede arbitrarily far away from the origin. Like the hyperbolic paraboloid, the hyperboloid of one sheet is a ruled surface.