

First Orderizing Second Order ODE and Phase Space

Writing a second order ordinary differential equation (ODE) as a system of first order ODE is a useful skill. There are analytical tools for studying first order systems, one of which we will look at in these notes. A first order presentation can also be useful in studying the equations numerically; for instance the Matlab function `ode45` takes an ODE in first order form. When we write an ODE in first order form we can relate it to concepts of vector calculus which we have been studying, vector fields and integral curves.

An example: Consider the following example

$$\frac{d^2 f(t)}{dt^2} = -k f(t). \quad (1)$$

This equation is often used in physics. For example $f(t)$ could be the displacement of a ball on a spring, and Eq. (1) says that the force of the spring on the ball is proportional to the displacement of the ball (Hook's law). It can also be used to describe a pendulum in the small angle approximation.

We now write this as a system of first order equations. The general method is to introduce new variables u^1 and u^2 defined by

$$u^1(t) = f(t), \quad u^2(t) = \frac{df(t)}{dt}. \quad (2)$$

The new variables can be considered as components of a vector $\bar{\mathbf{u}} = (u^1, u^2)$. The first component is just the original function, while the second component is the first derivative of the function. In terms of physical problem, $u^1(t)$ represents the position of the system (the ball on the spring for example) and $u^2(t)$ represents the velocity or momentum (the velocity of the ball as the spring oscillates). Note that each is in general changing in time.

To obtain a system of first order ODE for $(u^1(t), u^2(t))$ we differentiate each component, and use the variable definitions and the original equation. We find

$$\frac{du^1(t)}{dt} = \frac{df(t)}{dt} = u^2(t) \quad \text{and} \quad \frac{du^2(t)}{dt} = \frac{d^2 f(t)}{dt^2} = -k u^1(t).$$

Combining, we have

$$\frac{du^1(t)}{dt} = u^2(t), \quad (3)$$

$$\frac{du^2(t)}{dt} = -k u^1(t), \quad (4)$$

or in vector form

$$\frac{d}{dt} \bar{\mathbf{u}}(t) = \bar{\mathbf{F}}(\bar{\mathbf{u}}), \quad (5)$$

where $\bar{\mathbf{F}}(\bar{\mathbf{u}}) = (u^2, -k u^1)$.

Now, without solving the ODE we can learn about the behavior of the system by investigating the vector field $\bar{\mathbf{F}}$ in the u^1, u^2 - plane. Note that the solution to the system Eqs. (3)-(4) describes the integral curves (also called flow lines, or field lines) of the vector field $\bar{\mathbf{F}}(\bar{\mathbf{u}}) = (u^2, -k u^1)$. One can imagine starting at some point in the u^1, u^2 - plane, and following the vector field $\bar{\mathbf{F}}$ to another point in the u^1, u^2 - plane. A sketch of the vector field and couple integral curves is included below in Figure 1.

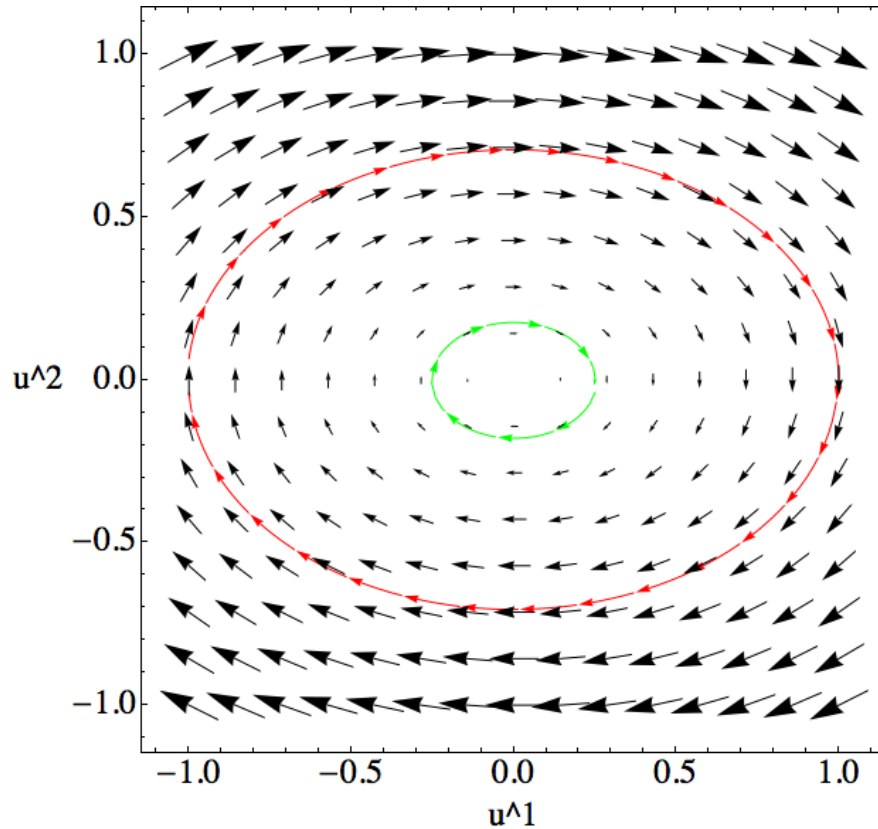


Figure 1: The vector field $\bar{\mathbf{F}} = (u^1, -ku^2)$, for $k = 1/2$, and a couple integral curves.

The u^1, u^2 - plane is a kind of state space, since any point in this plane represents the system with position u^1 and velocity u^2 . This is often called *phase space*, and the vector $\bar{\mathbf{F}}$ could be called the *phase vector field*. We see that if we start our ball on the spring with initial conditions, that is initial position and initial velocity, this corresponds to a particular point in phase space. Since the system is governed by the equations Eqs. (3)-(4) the systems state will follow the phase vector field along an integral curve. From the plot of the vector field we can tell that this system will oscillate, and in fact at a later time will return to the same state in which we started it! Of course we knew this intuitively for the ball on a spring, but that is the idea.

More Problems Here are some similar problems to which you can apply the same sort of analysis.

Problem 1 (Free Particle). Consider the equation of a free particle in one dimension

$$\frac{d^2x(t)}{dt^2} = 0.$$

Write as a system of first order ODE, sketch the phase space vector field, some integral curves, and interpret the behavior of the system using the vector field.

Problem 2 (Constant Acceleration). Consider the equation of a particle in one dimension under constant acceleration

$$\frac{d^2x(t)}{dt^2} = -g.$$

Write as a system of first order ODE, sketch the phase space vector field, some integral curves, and interpret the behavior of the system using the vector field.

Problem 3 (Inverted Pendulum). Consider the equation

$$\frac{d^2\phi(t)}{dt^2} = \phi(t).$$

Write as a system of first order ODE, sketch the phase space vector field, some integral curves, and interpret the behavior of the system using the vector field. How does this represent an inverted pendulum? How many separated regions are there in the phase space?