Exam for MVE041 och MMGL32 Flervariabelmatematik

The 28 August 2015, kl. 830-1230

Help materials: Attached formula sheet. No calculators.

Telephone: Frida Svelander, 0703-088304

Total points are 50. Passing this course requires a) 25 points of 32 points on the *Passing Part*, and b) a pass on all six Matlab labs. Your bonus points from this course apply to the passing part of the exam. The maximum score on the passing part is 32 including bonus points. A grade of 4 or 5 is obtained with scores of 33, and 42 respectivley. Bonus points do not apply to the mastery part of the exam.

Solutions will be posted on the course website on the first weekday following the exam. The exam is graded anonymously. Results are available on Ladok starting three weeks after the exam day. The first day on which you may contest your grade will be posted on the course website, and after that you may file a contest with the MV exp weekdays 9-13.

Passing Part

1. Consider the function $f(x,y) = 21x^2 - 7y$.
   (a) (4 pts) What is the derivative of $f(x,y)$ in the direction of the vector $\bar{v} = 2\hat{\imath} + 3\hat{\jmath}$ at the point $P = (\frac{1}{6}, 1)$?
   (b) (2 pts) For the graph $z = f(x,y)$, write the equation for the $z = 14$ level curve, and make a sketch.

2. (3 pts) Find all critical points of the function $f(x,y) = xy - x + y^2$.

3. (4 pts) Let $f(x,y) = x^3y + 3y^2$, and suppose $x = e^t, y = t^3$. Use the chain rule to compute $\frac{d}{dt}z(t) = \frac{d}{dt}f(x(t), y(t))$ and write as a function of t only.

4. (3 pts) Let C be the helical curve $\bar{r}(t) = 4\cos(t)\hat{\imath} + 4\sin(t)\hat{\jmath} + 3t\hat{k}$ from $t = 0$ to $t = 2\pi$. Compute the line integral $I = \int_C z\,ds$.

5. (4 pts) What is the volume of the region $x^2 + y^2 \leq z \leq 2 - \sqrt{x^2 + y^2}$?

6. Consider the vector field $\bar{G}(x,y) = \left(ye^{xy} + 3x^2y\right)\hat{\imath} + \left(xe^{xy} + x^3\right)\hat{\jmath}$.
   (a) (3 pts) Compute $\bar{\nabla} \cdot \bar{G}$ and $\bar{\nabla} \times \bar{G}$.
   (b) (3 pts) Find a function ϕ such that $\bar{G} = \bar{\nabla}\phi$ and compute $\int_C \bar{G} \cdot d\bar{r}$ for any curve C from $P = (0,5)$ to $Q = (1,2)$.

7. (6 pts) Write down the equation of Green's theorem. Let $\bar{F}(x,y) = \left(\cos(x)\sin(y) - \frac{y^2x^2}{4}\right)\hat{\imath} + \left(\sin(x)\cos(y) + \frac{yx^3}{6}\right)\hat{\jmath}$. Compute $\int_C \bar{F} \cdot d\bar{r}$ where C is the closed path bounding the triangle with corners $(0,0)$, $(1,1)$, and $(0,1)$ oriented counterclockwise.

Mastery Part

8. (6 pts) Integrate $f(x,y) = xy$ over the region in the plane defined by $y > x^2$, $y < 5x^2$, and $1 < xy < 4$.

9. Extreme Values with Constraints
   (a) (3 pts) Let C be curve defined by $g(x,y) = 0$, and let $P_0 = (x_0, y_0)$ be an interior point on C at which $\bar{\nabla}g$ is non-vanishing and at which $f(x,y)$ has a local extreme value. Prove that there exists a λ_0 such that (x_0, y_0, λ_0) is a critical point of $L(x,y,\lambda) = f(x,y) + \lambda g(x,y)$.
   (b) (3 pts) What are the maximum and minimum values of $f(x,y) = x + y - x^2 - 3y^2$ on the region defined by $x^2 + 3y^2 \leq 1$?

10. (6 pts) Compute the flux of $\bar{F} = y\left(\frac{x}{\sqrt{x^2+y^2}}\hat{\imath} + .\frac{y}{\sqrt{x^2+y^2}}\hat{\imath}\right)$ out of the graph of the function $f(x,y) = 1 - x^2 - y^2$ over the first quadrant. Does the divergence theorem apply in this case? Why or why not?

MVE041 20150828 Exam Solutions

Passing

1/ a) Want $\bar{u}.\bar{\nabla}f\big|_P$, $\bar{u} = \bar{V}/\|\bar{V}\|$, $P=(1/6, 1)$

$$\|\bar{V}\| = \sqrt{4+9} = \sqrt{13} \Rightarrow \bar{u} = \frac{1}{\sqrt{13}}(2\hat{i}+3\hat{j})$$

$$\partial_x f = 42x, \quad \partial_y f = -7 \Rightarrow \bar{\nabla}f = 42x\,\hat{i} - 7\hat{j}$$

$$\therefore \bar{u}.\bar{\nabla}f = \frac{84}{\sqrt{13}}x - \frac{21}{\sqrt{13}}$$

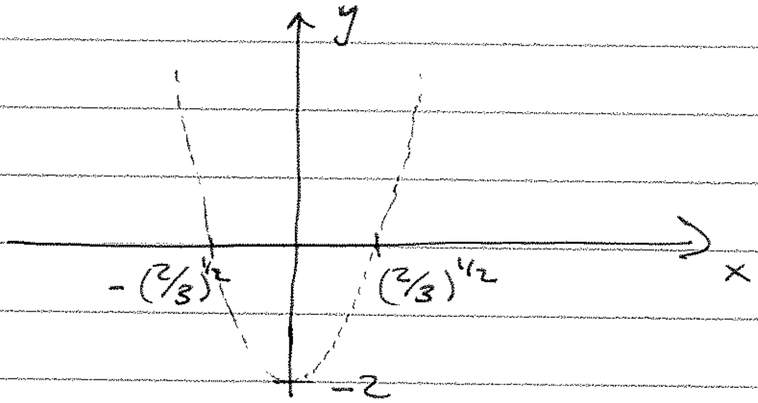
$$\bar{u}.\bar{\nabla}f\Big|_{P=(\frac{1}{6},1)} = \frac{84}{\sqrt{13}}\left(\frac{1}{6}\right) - \frac{21}{\sqrt{13}}$$

$$= \frac{1}{\sqrt{13}}(14-21)$$

$$= -7/\sqrt{13}$$

b) $z = f(x,y) = 14 \Leftrightarrow$ $14 = 21x^2 - 7y$

$$y = 3x^2 - 2$$



2/ Critical point : $\bar{\nabla} f(x_0, y_0) = 0$

$$\frac{\partial}{\partial x} f = y - 1 , \quad \frac{\partial f}{\partial y} = x + 2y$$

$$\bar{\nabla} f = (y-1)\hat{\imath} + (x+2y)\hat{\jmath}$$

$$\bar{\nabla} f = 0 \quad \Rightarrow \quad y = 1 , \quad x = -2y = -2$$

$$\boxed{(x_0, y_0) = (-2, 1)}$$

3/ Let $z(t) = f(x(t), y(t))$
where $x(t) = e^t$, $y = t^3$
and $f(x,y) = x^3 y + 3y^2$

Chain rule: $\dfrac{dz(t)}{dt} = \dfrac{\partial f}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dt}$

$$\frac{\partial f}{\partial x} = 3x^2 y = 3e^{2t} t^3$$

$$\frac{\partial f}{\partial y} = x^3 + 6y = e^{3t} + 6t^3$$

$$\frac{dx}{dt} = e^t , \quad \frac{dy}{dt} = 3t^2$$

$$\therefore \frac{dz}{dt} = 3e^{2t} t^3 e^t + (e^{3t} + 6t^3) 3t^2$$

$$\boxed{= 3t^2 e^{3t}(t+1) + 18t^5}$$

4/

$$I = \int_C z\,ds = \int_{t=0}^{2\pi} z(t)\,v(t)\,dt$$

where

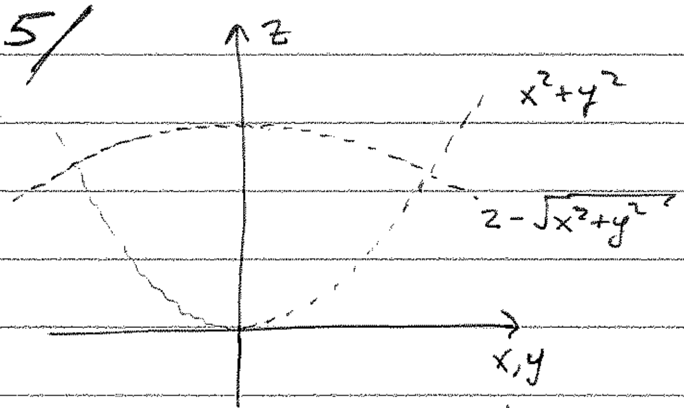
$$v(t) = \left|\frac{d\bar{r}}{dt}\right| = \sqrt{16+9} = 5$$

$$z(t) = 3t$$

$$I = \int_0^{2\pi} 15\,t\,dt = \frac{15}{2}\left[t^2\right]_0^{2\pi}$$

$$\boxed{I = 30\pi^2}$$

5/



Cylindrical coords (S, z, φ)

$$Vol = \iiint S\,dS\,dz\,d\varphi$$

$$= 2\pi \int_{S=0}^{S_{max}} \int_{z=S^2}^{2-S} S\,dz\,dS$$

$$x^2+y^2 = S^2$$

$$2-\sqrt{x^2+y^2} = 2-S$$

S_{max}: $S^2 = 2-S$

$$\Rightarrow (S+2)(S-1) = 0$$

$$S_{max} = 1$$

$$= 2\pi \int_{S=0}^{S_{max}} S\left(2-S-S^2\right) dS$$

$$= 2\pi \left[S^2 - \frac{1}{3}S^3 - \frac{1}{4}S^4\right]_0^1$$

$$= 2\pi\left(1 - \frac{1}{3} - \frac{1}{4}\right)$$

$$\therefore \boxed{Vol = \frac{5}{6}\pi}$$

(4)

b/ $\bar{G} = (y e^{xy} + 3x^2 y)\hat{i} + (x e^{xy} + x^3)\hat{j}$

a) $\bar{\nabla} \cdot \bar{G} = \dfrac{\partial G^1}{\partial x} + \dfrac{\partial G^2}{\partial y}$

$= y^2 e^{xy} + 6xy + x^2 e^{xy} + 0$

$$\boxed{= e^{xy}(x^2+y^2) + 6xy}$$

$\nabla \times \bar{G} = \left(\dfrac{\partial G^2}{\partial x} - \dfrac{\partial G^1}{\partial y}\right)\hat{k}$

$= e^{xy} + xy e^{xy} + 3x^2 - \left(e^{xy} + xy e^{xy} + 3x^2\right)$

$$\boxed{= 0}$$

b) $\dfrac{\partial \phi}{\partial x} = G^1 = y e^{xy} + 3x^2 y$

$\Rightarrow \phi(x,y) = \int (y e^{xy} + 3x^2 y)\,dx = e^{xy} + x^3 y + h(y)$

$\dfrac{\partial \phi}{\partial y} = G^2 = x e^{xy} + x^3$

$\Rightarrow \phi(x,y) = \int (x e^{xy} + x^3)\,dy = e^{xy} + x^3 y + g(x)$

$\therefore h(y) = g(x) = c$ ~ constant

$$\boxed{\phi(x,y) = e^{xy} + x^3 y + c}$$

$\int_c \bar{G} \cdot d\bar{r} = \int_c \bar{\nabla}\phi \cdot d\bar{r} = \phi(1,2) - \phi(0,5)$

$$\boxed{= e^2 + 1}$$

7/

$$\oint_C \bar{F}\cdot d\bar{r} = \pm \iint_R \left(\frac{\partial F^2}{\partial x} - \frac{\partial F^1}{\partial y}\right) dA$$

"+" counter-clockwise, "−" clockwise

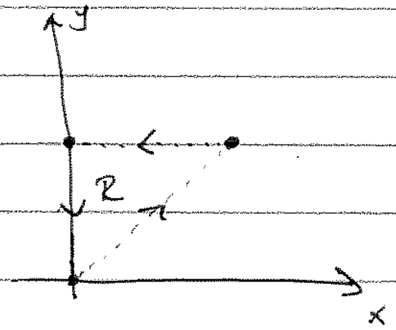
or

$$\oint_C \bar{F}\cdot d\bar{r} = \iint_R \text{curl}\,\bar{F}\cdot\hat{k}\, dA$$

$$|\text{curl}\,\bar{F}| = \frac{\partial F^2}{\partial x} - \frac{\partial F^1}{\partial y} = \cos(x)\cos(y) + \frac{1}{2}yx^2 - \left(\cos(x)\cos(y) - \frac{1}{2}yx^2\right)$$

$$= yx^2$$

counter clockwise



$$\oint_C \bar{F}\cdot d\bar{r} = +\iint_R yx^2\, dA$$

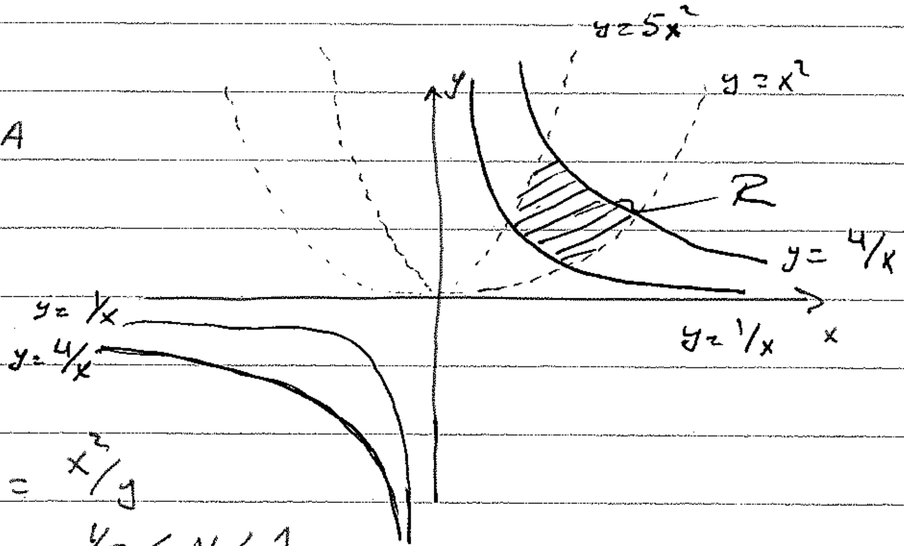
$$= \int_{y=0}^{1} y \int_{x=0}^{y} x^2\, dx\, dy$$

$$= \int_{y=0}^{1} \frac{1}{3} y\, y^3$$

$$\boxed{= \frac{1}{15}}$$

Mastery

8/ $$I = \iint_R xy \, dA$$



Let $u = xy$, $v = x^2/y$

$1 < u < 4$, $1/5 < v < 1$

$$\left|\frac{\partial(u,v)}{\partial(x,y)}\right| = \det\begin{pmatrix} y & x \\ 2\frac{x}{y} & -\frac{x^2}{y^2}\end{pmatrix} = 3\,x^2/y$$

and $$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \left|\frac{\partial(u,v)}{\partial(x,y)}\right|^{-1} = \frac{1}{3v}$$

$$I = \iint_R xy \, dx\,dy = \iint_D u \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

$$= \frac{1}{3}\int_{v=1/5}^{1}\int_{u=1}^{4} \frac{u}{v}\, du\, dv$$

$$= \frac{1}{3}\,\frac{1}{2}(16-1)\,(\log(1) - \log(1/5))$$

$$\boxed{= \frac{15}{6}\log(5)} = \boxed{\frac{5}{2}\log(5)}$$

9/

a) Critical point (x_0, y_0, λ_0) of $L(x,y,\lambda) = f(x,y) + \lambda g(x,y)$

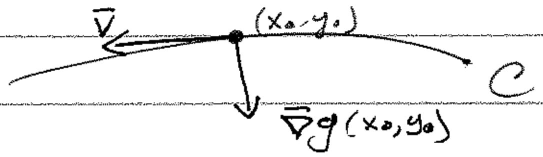
$\Longleftrightarrow$

A) $\dfrac{\partial L}{\partial x} = \dfrac{\partial f}{\partial x}\Big|_{(x_0,y_0)} + \lambda_0 \dfrac{\partial g}{\partial x}\Big|_{(x_0,y_0)} = 0$

B) $\dfrac{\partial L}{\partial y} = \dfrac{\partial f}{\partial y}\Big|_{(x_0,y_0)} + \lambda_0 \dfrac{\partial g}{\partial y}\Big|_{(x_0,y_0)} = 0$

C) $\dfrac{\partial L}{\partial \lambda} = g(x,y)\Big|_{(x_0,y_0)} = 0$

Note (C) is satisfied at (x_0, y_0) by definition of C.



The gradient of g is orthogonal to the level curve $g = 0$ at (x_0, y_0).

Note that if $\bar{\nabla} f \parallel \bar{\nabla} g$ *, then $\exists\ \lambda_0$ such that $\bar{\nabla} f\Big|_{(x_0,y_0)} = -\lambda_0 \bar{\nabla} g\Big|_{(x_0,y_0)}$, and

* at (x_0, y_0)

(A), (B) are satisfied.

Suppose now that $\bar{\nabla} f\Big|_{(x_0,y_0)}$ is not $\parallel$ to $\bar{\nabla} g\Big|_{(x_0,y_0)}$. Then $\bar{v} \cdot \bar{\nabla} f\Big|_{(x_0,y_0)} \neq 0$, where $\bar{v}$ is the tangent vector to C at (x_0, y_0). But this would mean that f is increasing or decreasing in the direction of $\bar{v}$, and thus (x_0, y_0) is not a local extremum of f. Therefore $\bar{\nabla} f \parallel \bar{\nabla} g$, and we have the above case. $\square$

9/ b)

Critical Points in interior:

$$\bar{\nabla} f = (1-2x)\hat{i} + (1-6y)\hat{j}$$

$\Longrightarrow$ $(x_0, y_0) = \left(\frac{1}{2}, \frac{1}{6}\right)$ is a critical point.

Further $f(x_0,y_0) = \frac{1}{2} + \frac{1}{6} - \frac{1}{4} - \frac{3}{36} = \frac{1}{4} + \frac{3}{36} = \frac{1}{3}$

Extremum Values on bdry: Lagrange Multipliers

Let $g(x,y) = x^2 + 3y^2 - 1$

and

$$L(x,y,\lambda) = f(x,y) + \lambda g(x,y)$$

$$\left.\begin{aligned} \frac{\partial L}{\partial x} &= 1 - 2x + \lambda 2x = 0 \\ \frac{\partial L}{\partial y} &= 1 - 6y + \lambda 6y = 0 \end{aligned}\right\} \Longrightarrow 2x = 6y$$

$$\frac{\partial L}{\partial \lambda} = x^2 + 3y^2 - 1 = 0$$

$\therefore \quad 12y^2 = 1 \iff y = \pm 1/\sqrt{12} = \pm \sqrt{3}/6$

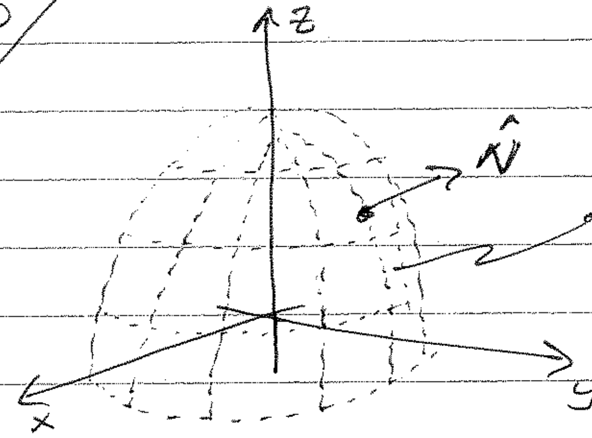
So we get two points $\underset{B_+}{\left(\sqrt{3}/2, \sqrt{3}/6\right)}$ and $\underset{B_-}{\left(-\sqrt{3}/2, -\sqrt{3}/6\right)}$

$$f|_{B_+} = \frac{2}{3}\sqrt{3} - 1, \quad f|_{B_-} = -\frac{2}{3}\sqrt{3} - 1$$

Can show $\frac{2}{3}\sqrt{3} - 1 < \frac{1}{3}$. Maximum $\frac{1}{3}$ at $\left(\frac{1}{2}, \frac{1}{6}\right)$

Minimum of $-\frac{2}{3}\sqrt{3} - 1$ at $\left(-\frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{6}\right)$

10/



S the graph of $f(x,y) = 1 - x^2 - y^2$ over first quadrant, Q.

$$\text{Flux: } \Phi = \iint_S \bar{F} \cdot d\bar{S} = \iint_S \bar{F} \cdot \hat{N}\, dS$$

$$\bar{F} = \frac{y}{\sqrt{x^2+y^2}} \left(x\hat{\imath} + y\hat{\jmath} + 0\hat{k} \right)$$

$$\hat{N}\, dS = \left(-\frac{\partial f}{\partial x}\hat{\imath} - \frac{\partial f}{\partial y}\hat{\jmath} + \hat{k} \right) dx\,dy$$

$$\cancel{\therefore \Phi =} \qquad \frac{\partial f}{\partial x} = -2x \,, \quad \frac{\partial f}{\partial y} = -2y$$

$$\therefore \Phi = \iint_Q \frac{y}{\sqrt{x^2+y^2}} \left(2x^2 + 2y^2 \right) dx\,dy$$

$$= 2\iint_Q y\sqrt{x^2+y^2}\, dx\,dy$$

// Polar coords $x = s\cos\theta$, $y = s\sin\theta$
$dx\,dy = s\,ds\,d\theta$ //

$$= 2\int_0^{\pi/2}\int_{s=0}^{1} s \cdot \sin\theta \cdot s \cdot s\, ds\, d\theta$$

$$= \frac{2}{4}\int_0^{\pi/2} \sin\theta\, d\theta$$

$$\boxed{= 1/2}$$