Tentamen i MVE041 Flervariabelmatematik

Solutions

Den 26 August 2016, kl. 830-1230
Hjälpmedel: Formelblad (bifogat), inga räknare.
Telefon: Adam Malik, 5325

Tentamen består av en gödkantdel (38 poäng) och överbetygsdel (12 poäng), sammanlagt 50 poäng. För godkänt på tentamen krävs 25 poäng på gödkantdel. För betyg 4 krävs totalt 33 poäng var av minst 4 på överbetygsdel. För betyg 5 krävs totalt 43 poäng var av minst 7 på överbetygsdel. Bonuspoäng från 2016 räknas in i totala poängen. För att bli godkänd på hela kursen måsta du klara tentamen och alla 6 Matlab laborationer.

Lösningar läggs ut på kursens webbsida första vardagen efter tentamensdagen. Tentan rättas och bedöms anonymt. Resultat meddelas via Ladok ca. tre veckor efter tentamenstillfället. Första granskningstillfälle meddelas på kurswebbsidan, efter detta sker granskning alla vardagar 9-13, MV:s exp.

Godkäntdel

1. Betrakta sfären med radie a centrerad i origo.
   (a) (1 p) Om hela sfären beskrivs som nivåytan $G(x,y,z)=0$, vad är funktionen $G(x,y,z)$?
   (b) (2 p) Skriv undre halv-sfär som grafen $z=f(x,y)$ av en funktion $f(x,y)$. Vad är domänen för $f(x,y)$?
   (c) (3 p) Vad är ekvationen till tangentplanet för undre halv-sfär när $(x,y)=(\frac{1}{3}a, \frac{2}{3}a)$?

2. (5 p) Anta att du löser systemet $f_1(x,y)=0, f_2(x,y)=0$ med Newtons metod, då
$$f_1(x,y)=e^{xy}+x^2+y^3-5, \quad f_2(x,y)=2\cos(x)\sin(y).$$
Skriv en ekvation för k^{th} approximation till lösningen.

3. (5 p) Betrakta kurvan på paraboloiden $f(x,y)=x^2+y^2$ beskrivs av $z(t)=f(x(t),y(t))$, var $x(t)=t^2\cos(t)$, och $y(t)=t^2\sin(t)$. Vad är farten $\frac{dz(t)}{dt}$ vid vilken höjden z förändringar längs denna stig? Skriva som en funktion av t.

4. (5 p) Hitta volymen mellan graferna $z=x^2+y^2$ och $z=2-\sqrt{x^2+y^2}$.

5. (6 p) Integrera funktionen $f(x,y)=(2x+6)/y$ längs parabeln $x-y^2+3=0$ från $(-2,-1)$ till $(1,2)$.

6. (5 p) Hitta en potentiel $\phi(x,y)$ så att $\bar{\mathbf{F}}(x,y)=\bar{\nabla}\phi(x,y)$ då $\bar{\mathbf{F}}(x,y)=y^3e^{y^2x}\hat{\imath}+(e^{y^2x}(1+2y^2x)+3)\hat{\jmath}$. Beräkna $\int_C \bar{\mathbf{F}}(x,y)\cdot d\bar{\mathbf{r}}$, för godtycklig kurva C från $(0,5)$ till $(\ln(2),1)$.

7. (6 p) Beräkna flödet av $\bar{\mathbf{V}}(x,y,z)=(3x^2+y^3)\hat{\imath}+4xz\hat{\jmath}+z^3\hat{\mathbf{k}}$ ut av sfären med radie a med hjälp av Guass sats.

Överbetygsdel

8. (4 p) Om $\bar{\mathbf{F}}(x,y)=(x^2y-8y+\frac{3}{4}xy^2)\hat{\imath}+(\frac{3}{4}x^2y+8x-\frac{16}{9}xy^2)\hat{\jmath}$, över vilken slutna kurva C är integralen $\oint_C \bar{\mathbf{F}}(x,y)\cdot d\bar{\mathbf{r}}$ minimerad?

9. (4 p) Bestäm aren av den sfären $x^2+y^2+z^2=4$, som ligger inuti cylindern $x^2+y^2=2x$.

10. (4 p) Låt $f(x,y)$ och $g(x,y)$ vara släta funktioner, och låt $P_0=(x_0,y_0)$ var en punkt på kurvan $g(x,y)=0$ där $\bar{\nabla}g$ är noll-skild och $f(x,y)$ har lokalt extremvärde. Bevisa att det finns en λ_0 så att (x_0,y_0,λ_0) är en kritisk punkt till $L(x,y,\lambda)=f(x,y)+\lambda g(x,y)$.

Exam for MVE041 Flervariabelmatematik

26 August 2016, kl. 830-1230

Help materials: Attached formula sheet. No calculators.

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The exam consists of a passing section (38 points) and mastery section (12 points), for a total of 50 points. A passing grade on the exam is obtained with a score of 25 points from the passing part. A grade 4 is obtained with 33 points in total, and at least 4 from the mastery section. A grade 5 is obtained with 43 points in total, and at least 7 from the mastery section. Bonus points earned during the 2016 course are included in the total scores. To pass the entire course you must pass the exam and complete all six laborations.

Solutions will be posted on the course website on the first weekday following the exam. The exam is graded anonymously. Results are available on Ladok starting three weeks after the exam day. The first day on which you may contest your grade will be posted on the course website, and after that you may file a contest with the MV exp weekdays 9-13.

Passing Part

1. Consider the sphere of radius a centered at the origin.
   (a) (1 p) If the entire sphere is described as the level surface $G(x, y, z) = 0$, what is the function $G(x, y, z)$?
   (b) (2 p) Write the lower hemisphere as the graph $z = f(x, y)$ of a function $f(x, y)$. What is the domain of $f(x, y)$?
   (c) (3 p) What is the equation for the tangent plane to the lower hemi-sphere when $(x, y) = (\frac{1}{3}a, \frac{2}{3}a)$?

2. (5 p) Suppose you are solving the system $f_1(x, y) = 0, f_2(x, y) = 0$ using Newton's method, where
$$f_1(x, y) = e^{xy} + x^2 + y^3 - 5, \quad f_2(x, y) = 2\cos(x)\sin(y).$$
Write down an equation for the k^{th} approximation to the solution.

3. (5 p) Consider the path on the paraboloid $f(x, y) = x^2 + y^2$ described by $z(t) = f(x(t), y(t))$, where $x(t) = t^2\cos(t)$, and $y(t) = t^2\sin(t)$. What is the speed $\frac{dz(t)}{dt}$ at which the height z changes along this path? Write as a function of t.

4. (5 p) Find the volume between the graphs $z = x^2 + y^2$ and $z = 2 - \sqrt{x^2 + y^2}$.

5. (6 p) Integrate the function $f(x, y) = (2x + 6)/y$ along the parabola $x - y^2 + 3 = 0$ from $(-2, -1)$ to $(1, 2)$.

6. (5 p) Find a potential $\phi(x, y)$ such that $\bar{\mathbf{F}}(x, y) = \bar{\nabla}\phi(x, y)$ where $\bar{\mathbf{F}}(x, y) = y^3 e^{y^2 x}\hat{\imath} + (e^{y^2 x}(1 + 2y^2 x) + 3)\hat{\jmath}$. Compute $\int_C \bar{\mathbf{F}}(x, y) \cdot d\bar{\mathbf{r}}$, for any curve C from $(0, 5)$ to $(\ln(2), 1)$.

7. (6 p) Compute the flux of $\bar{\mathbf{V}}(x, y, z) = (3x^2 + y^3)\hat{\imath} + 4xz\hat{\jmath} + z^3\hat{\mathbf{k}}$ out of the sphere of radius a using the divergence theorem.

Mastery Part

8. (4 p) If $\bar{\mathbf{F}}(x, y) = (x^2y - 8y + \frac{3}{4}xy^2)\hat{\imath} + (\frac{3}{4}x^2y + 8x - \frac{16}{9}xy^2)\hat{\jmath}$, over what closed curve C is the integral $\oint_C \bar{\mathbf{F}}(x, y) \cdot d\bar{\mathbf{r}}$ minimized?

9. (4 p) Find the surface area of the sphere $x^2 + y^2 + z^2 = 4$, which lies inside the cylinder $x^2 + y^2 = 2x$.

10. (4 p) Let $f(x, y)$ and $g(x, y)$ be smooth functions, and let $P_0 = (x_0, y_0)$ be a point on the curve $g(x, y) = 0$ at which $\bar{\nabla}g$ is non-vanishing and at which $f(x, y)$ has a local extreme value. Prove that there exists a λ_0 such that (x_0, y_0, λ_0) is a critical point of $L(x, y, \lambda) = f(x, y) + \lambda g(x, y)$.

Exam Solutions 160826

1/ a) $G(x,y,z) = x^2+y^2+z^2-a^2$

b) $z = f(x,y) = -\sqrt{a^2-x^2-y^2}$

$D(f) = \{(x,y)\in\mathbb{R}^2 \mid x^2+y^2 \leq a^2\}$

c) $z = f(x_0,y_0) + f_1(x_0,y_0)(x-x_0) + f_2(x_0,y_0)(y-y_0)$

$f(a/3, 2a/3) = -\sqrt{a^2-\frac{a^2}{9}-4\frac{a^2}{9}} = -\frac{2}{3}a$

$f_1 = x/\sqrt{\ldots} \Rightarrow f_1(a/3, 2a/3) = \frac{a/3}{2a/3} = 1/2$

$f_2 = y/\sqrt{\ldots} \Rightarrow f_2(a/3, 2a/3) = \frac{2a/3}{2/3 a} = 1$

$$\boxed{\therefore\ z = -\frac{2}{3}a + \frac{1}{2}(x-a/3) + 1\left(y-\frac{2}{3}a\right)}$$

2/ At each Newton step we solve for $\begin{pmatrix}x_k\\y_k\end{pmatrix}$ given by

$$\begin{pmatrix}x_k\\y_k\end{pmatrix} = \begin{pmatrix}x_{k-1}\\y_{k-1}\end{pmatrix} - \left[D\bar{F}(x_{k-1},y_{k-1})\right]^{-1}\bar{F}(x_{k-1},y_{k-1})$$

where $\bar{F} = \begin{pmatrix}f_1\\f_2\end{pmatrix}$ and $D\bar{F} = \begin{pmatrix}\frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y}\\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y}\end{pmatrix}$

$$D\bar{F}(x,y) = \begin{pmatrix} ye^{xy}+2x & xe^{xy}+3y^2\\ -2\sin(x)\sin(y) & 2\cos(x)\cos(y)\end{pmatrix}$$

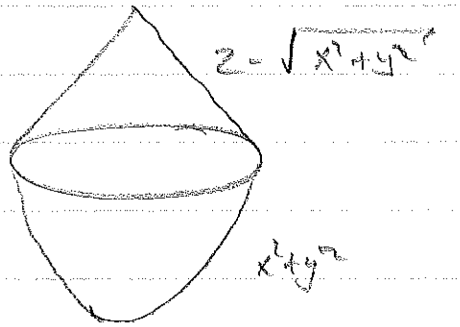
3/ $$\frac{dz(t)}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

$$= 2x\left(2t\cos(t) - t^2\sin(t)\right) + 2y\left(2t\sin(t) + t^2\cos(t)\right)$$

$$= 4t^3\cos^2 t - \cancel{2t^4\cos t\sin t}$$

$$+ 4t^3\sin^2 t + \cancel{2t^4\cos t\sin t}$$

$$\boxed{= 4t^3}$$

4/ 

$$V = \int_0^{2\pi}\int_{r=0}^{r_m}\int_{z=r^2}^{2-r} r\,dz\,dr\,d\theta$$

$$= 2\pi\int_0^1 r\left(2-r-r^2\right)dr$$

$$= 2\pi\left(1 - 1/3 - 1/4\right)$$

$$\Rightarrow \boxed{V = \frac{5}{6}\pi}$$

Find r_m:

$2-r=r^2$

$\Rightarrow r^2+r-2 = (r+2)(r-1) = 0$

$\Rightarrow r_m = 1$

5/

$$I = \int f(\bar{r}(t))\, v(t)\, dt$$

Parametrize by y: $\bar{r}(t) = (t^2-3)\hat{i} + t\hat{j}$, $t \in (-1,2)$

$\Rightarrow \bar{v}(t) = 2t\hat{i} + \hat{j}$, $v(t) = \sqrt{4t^2+1}$

$$\therefore \quad I = \int_{-1}^{2} \frac{2(t^2-3)+6}{t} \sqrt{4t^2+1}\, dt$$

$$= \int_{-1}^{2} 2t\sqrt{4t^2+1}\, dt$$

// $u = 4t^2+1$, $du = 8t\, dt$

$t=-1 \rightsquigarrow u=5$, $t=2 \rightsquigarrow u=17$ //

$$= \frac{1}{4}\int_{5}^{17} \sqrt{u}\, du$$

$$= \frac{1}{4}\cdot\frac{2}{3}\left(17\sqrt{17} - 5\sqrt{5}\right)$$

$$\boxed{= \frac{1}{6}\left(17\sqrt{17} - 5\sqrt{5}\right)}$$

6/ $\phi(x,y) = \int \vec{F}\,dx = \int y^3 e^{y^2 x}\,dx$

$= \int \frac{d}{dx}\left(y\, e^{y^2 x}\right) dx$

$= y e^{y^2 x} + c(y)$

Then, $\frac{\partial \phi}{\partial y} = e^{y^2 x} + 2y^2 x e^{y^2 x} + \frac{\partial c(y)}{\partial y} = e^{y^2 x}(1 + 2y^2 x) + 3$

$\Rightarrow c(y) = 3y$

$\boxed{\therefore \quad \phi(x,y) = y e^{y^2 x} + 3y + C}$

$\int_C \vec{F}(\vec{r}) \cdot d\vec{r} = \phi(\ln(2), 1) - \phi(0, 5)$

$= 2 + 3 - (5 + 15)$

$\boxed{= -15}$

7/ $\bar{V} = (3x^2 + y^3)\hat{i} + 4xz\hat{j} + z^3\hat{k}$

$\Phi = \oiint_{S} \bar{V}\cdot\hat{N}\,dS = \iiint_{B} \operatorname{div}\bar{V}\,dV$, S is sphere rad. a

$\operatorname{div}\bar{V} = 6x + 0 + 3z^2$

$\iiint_{\text{Sphere}} (6x + 3z^2)\,dx\,dy\,dz = 3\iiint z^2\,dV$, by symmetry

$= 3\iiint \rho^2\cos^2\varphi\;\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta$

$= 6\pi \int_{\rho=0}^{a}\int_{\varphi=0}^{\pi} \rho^4\cos^2\varphi\sin\varphi\,d\rho\,d\varphi$

$= \frac{6\pi}{5}a^5 \int_{\varphi=0}^{\pi} \cos^2\varphi\sin\varphi\,d\varphi$

// $u = \cos\varphi$, $du = -\sin\varphi\,d\varphi$ $\quad \varphi = 0 \Rightarrow \cos\varphi = 1$

$\varphi = \pi \Rightarrow \cos\varphi = -1$ //

$= \frac{6\pi}{5}a^5 \int_{-1}^{1} u^2\,du$

$= \frac{2\pi}{5}a^5\; u^3\Big|_{-1}^{1} \quad \boxed{= \frac{4}{5}\pi a^5}$

$1 - (-1) = 2$

e/ $\bar{F} = \left(x^2y - 8y + \frac{3}{4}xy^2\right)\hat{i} + \left(\frac{3}{4}x^2y + 8x - \frac{16}{9}xy^2\right)\hat{j}$

$$\oint_C \bar{F}\cdot d\bar{r} = \iint_R \left(\frac{\partial F^2}{\partial x} - \frac{\partial F^1}{\partial y}\right) dA \quad , \text{ by Green's Th'm.}$$

$$= \iint_R \left[\cancel{\tfrac{3}{2}xy} + 8 - \frac{16}{9}y^2 - \left(x^2 - 8 + \cancel{\tfrac{3}{2}xy}\right)\right] dA$$

$$= \underbrace{\iint_R \left(16 - \frac{16}{9}y^2 - x^2\right) dx\,dy}$$

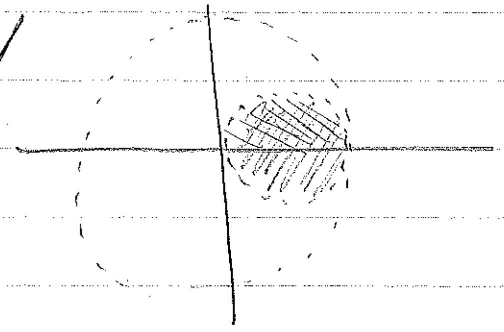
is min. if $f(x,y) = 16 - \frac{16}{9}y^2 - x^2$
is < 0 inside R and > 0 outside of R

$\Rightarrow f(x,y) = 0$ on $C = \text{bdry}(R)$

C is ellipse $16 - \frac{16}{9}y^2 - x^2 = 0$

$$\Rightarrow 1 = \frac{x^2}{4^2} + \frac{y^2}{3^2}$$

(7)

9/ 

Represent "top" portion of sphere by graph $z = f(x,y) = \sqrt{4 - x^2 - y^2}$

$$\iint_{top} dS = \iint_{top} |\bar{N}|\, dx\, dy, \quad \text{for a graph}$$

Outward normal: $\bar{N} = -\dfrac{-x}{\sqrt{\cdots}}\hat{i} - \dfrac{-y}{\sqrt{\cdots}}\hat{j} + \hat{k}$

so $|\bar{N}| = \left[\dfrac{x^2}{4-x^2-y^2} + \dfrac{y^2}{4-x^2-y^2} + 1\right]^{1/2} = \left[\dfrac{1}{4-x^2-y^2}\right]^{1/2}$

Let A be total surface area sought. Then in polar coords

$$\frac{1}{4}A = \iint_{top} \frac{r}{\sqrt{4-r^2}}\, dr\, d\theta = \int_{\theta=0}^{\pi/2} \int_{r=0}^{2\cos\theta} \frac{r}{\sqrt{4-r^2}}\, dr\, d\theta$$

// Note $x^2 + y^2 - 2x = 0 \Rightarrow r^2 - 2r\cos\theta = 0$
$\Rightarrow r(r - 2\cos\theta) = 0$ //

Thus $A = 4 \displaystyle\int_{\theta=0}^{\pi/2} \int_{r=0}^{2\cos\theta} \frac{r}{\sqrt{4-r^2}}\, dr\, d\theta$

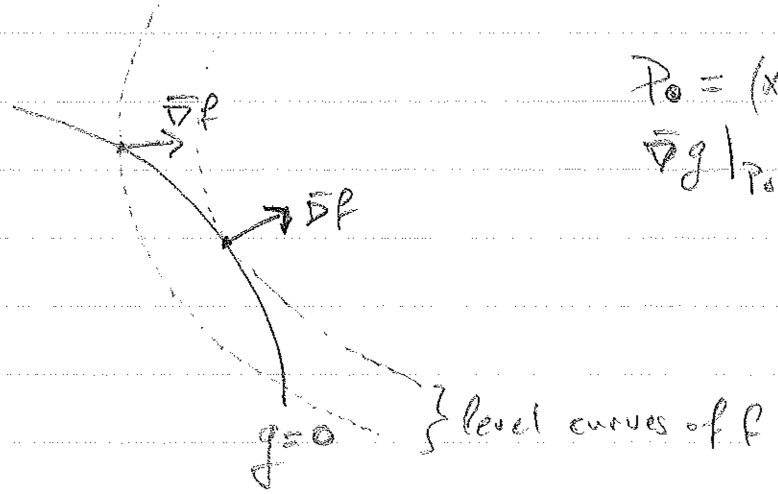
$$= -2 \int_{\theta=0}^{\pi/2} \left(2\sqrt{4-r^2}\right)\Big|_{r=0}^{2\cos\theta} d\theta$$

$$= -4 \int_{\theta=0}^{\pi/2} (2\sin\theta - 2)\, d\theta$$

$$= -4\,(2 - \pi)$$

$$\boxed{= -8 + 4\pi}$$

10/



$P_0 = (x_0, y_0)$ is on $g(x,y)=0$
$\bar{\nabla} g|_{P_0} \neq 0$

If (x_0, y_0, λ_0) is a critical point of $L = f + \lambda g$
Then

(1) $\bar{\nabla} f|_{P_0} = -\lambda_0 \bar{\nabla} g|_{P_0}$

(2) $\frac{\partial L}{\partial \lambda}\Big|_{\lambda_0, P_0} = 0$

Note that (2) is true by assumption $\frac{\partial L}{\partial \lambda}|_{P_0} = g|_{P_0} = 0$.

Suppose (1) is not true. Let $\bar{r}(t)$ be a parameterization of $g(x,y)=0$ near P_0 with velocity $\bar{v}(t)$. Under the assumption that (1) is not true
$\bar{\nabla} f|_{P_0} \cdot \bar{v}(t) = D_{\bar{v}} f(x_0, y_0) \neq 0$.
So that f is increasing or decreasing in the direction of $\bar{v}$ at P_0, which implies that P_0 is not a local extreme value for f. This is a contradiction.

Thus $\exists \lambda_0$ such that $\bar{\nabla} f|_{P_0} = -\lambda_0 \bar{\nabla} g|_{P_0}$

$\square$