Tentamen i MVE041 Flervariabelmatematik

Den 8 oktober 2016, kl. 830-1230

Hjälpmedel: Formelblad (bifogat), inga räknare.

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Solutions

Tentamen består av en gödkantdel (38 poäng) och överbetygsdel (12 poäng), sammanlagt 50 poäng. För godkänt på tentamen krävs 25 poäng på gödkantdel. För betyg 4 krävs totalt 33 poäng var av minst 4 på överbetygsdel. För betyg 5 krävs totalt 43 poäng var av minst 7 på överbetygsdel. Bonuspoäng från 2016 räknas in i totala poängen. För att bli godkänd på hela kursen måsta du klara tentamen och alla 6 Matlab laborationer.

Lösningar läggs ut på kursens webbsida första vardagen efter tentamensdagen. Tentan rättas och bedöms anonymt. Resultat meddelas via Ladok ca. tre veckor efter tentamenstillfället. Första granskningstillfälle meddelas på kurswebbsidan, efter detta sker granskning alla vardagar 9-13, MV:s exp.

Godkäntdel

1. Betrakta funktionen $f(x,y) = x^3 - 10x + y^2 - 6$.
   (a) (2 p) Bestäm gradienten av f.
   (b) (2 p) Hitta den utåtriktad enhets-normalvektorn till grafen $z = f(x,y)$ i punkten $P_0 = (2, 1, f(2,1))$.
   (c) (2 p) Ange en parametrisering för linjesegmentet $\bar{\mathbf{r}}(t)$ som börjar i P_0 vid $t = 0$, och vid $t = 1$ är en enhet i riktningen för normalvektorn.

2. (5 p) Vad är andra ordningens Taylor polynom för funktionen av två variabler $f(x,y)$ kring $(0,0)$? Med hjälp av denna allmänna formel, beräkna andra ordningens Taylor polynom för funktionen $f(x,y) = e^{3y-x}$ på $(0,0)$.

3. (6 p) Hitta alla möjliga extremevärden av $f(x,y) = 2xy - x - \frac{1}{2}y^2$ på triangel med hörn i $(0,0), (0,1), (1,1)$.

4. (5 p) Beräkna integralen $\iiint_T z dV$, där T är tetraedern i första oktanten som begränsas av koordinatplanen och planet $x + y + z = 1$.

5. (5 p) Bestäm divergensen och curl av $\bar{\mathbf{V}} = xe^{yz}\hat{\imath} + ye^{xz}\hat{\jmath} + ze^{xy}\hat{\mathbf{k}}$.

6. (6 p) $\bar{\mathbf{r}}(t) = 3\sin(2t)\hat{\imath} + (\cos(2t) + \cos(4t))\hat{\jmath}$ spårar ut en sluten kurva i medurs riktning som t går från 0 till π. Med hjälp av Greens sats bestäm arean av området som begränsas av denna kurva. *Tips: använd vektrofält* $\bar{\mathbf{F}} = -y\hat{\imath}$

7. (5 p) Bestäm flödet av $\bar{\mathbf{V}} = (xz+2)\hat{\imath} + yz\hat{\jmath} + xyz\hat{\mathbf{k}}$ ut ur sidorna (inte ändarna) av cylindern med radien a och ligger mellan $0 \le z \le h$.

Överbetygsdel

8. (4 p) Parametrisera integralkurvan för $\bar{\mathbf{V}} = -y\hat{\imath} + x\hat{\jmath} + c\hat{\mathbf{k}}$ som börjar vid $(1,0,0)$ genom sin båglängd.

9. (4 p) Betrakta riktningsderivatan definieras av

$$D_{\bar{\mathrm{u}}}f(a,b) := \frac{d}{dt}\bigg|_{t=0} f(a + tu_1, b + tu_2)$$

vid en punkt då $f(x,y)$ är differentierbar och $\bar{\nabla} f(a,b) \neq 0$. Vad är störst och mist värdena av $D_{\bar{\mathrm{u}}}f(a,b)$? Betyder $D_{\bar{\mathrm{u}}}f(a,b)$ försvinna? Om så är fallet, i vilka riktinger? Stöd dina svar.

10. (4 p) Använda divergens-satsen för att bestämma flödet av vektorfältet $\bar{\mathbf{V}} = (x + xz)\hat{\imath} + (x + yz)\hat{\jmath} - (2x + z^2)\hat{\mathbf{k}}$ ut ur den första oktant av sfären $x^2 + y^2 + z^2 = a^2$.

Exam for MVE041 Flervariabelmatematik

8 oktober 2016, kl. 830-1230

Help materials: Attached formula sheet. No calculators.

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The exam consists of a passing section (38 points) and mastery section (12 points), for a total of 50 points. A passing grade on the exam is obtained with a score of 25 points from the passing part. A grade 4 is obtained with 33 points in total, and at least 4 from the mastery section. A grade 5 is obtained with 43 points in total, and at least 7 from the mastery section. Bonus points earned during the 2016 course are included in the total scores. To pass the entire course you must pass the exam and complete all six laborations.

Solutions will be posted on the course website on the first weekday following the exam. The exam is graded anonymously. Results are available on Ladok starting three weeks after the exam day. The first day on which you may contest your grade will be posted on the course website, and after that you may file a contest with the MV exp weekdays 9-13.

Passing Part

1. Consider the function $f(x,y) = x^3 - 10x + y^2 - 6$.
   (a) (2 p) Compute the gradient vector of f.
   (b) (2 p) Find the outward pointing unit normal vector to the graph of f at the point $P_0 = (2, 1, f(2,1))$.
   (c) (2 p) Parametrize the line segment $\bar{\mathbf{r}}(t)$ which starts at P_0 at $t = 0$, and at $t = 1$ is a unit distance in the direction of the normal vector.

2. (5 p) What is second order Taylor's polynomial for a function of two variables $f(x,y)$ about the point $(0,0)$? Using this general formula, compute the second order Taylor polynomial for the function $f(x,y) = e^{3y-x}$ about $(0,0)$.

3. (6 p) Find all possible extreme values of $f(x,y) = 2xy - x - \frac{1}{2}y^2$ on the triangle with corners $(0,0), (0,1), (1,1)$.

4. (5 p) Evaluate the integral $\iiint_T z\,dV$, where T is the tetrahedron in the first octant which is bounded by the coordinate planes and the plane $x + y + z = 1$.

5. (5 p) Compute the divergence and the curl of $\bar{\mathbf{V}} = xe^{yz}\hat{\imath} + ye^{xz}\hat{\jmath} + ze^{xy}\hat{\mathbf{k}}$.

6. (6 p) $\bar{\mathbf{r}}(\mathbf{t}) = 3\sin(2t)\hat{\imath} + (\cos(2t) + \cos(4t))\hat{\jmath}$ traces out a closed curve in the clockwise direction as t goes from 0 to π. Using Green's theorem compute the area enclosed by this curve. *Hint: use the vector field* $\bar{\mathbf{F}} = -y\hat{\imath}$

7. (5 p) What is the flux of the vector field $\bar{\mathbf{V}} = (xz+2)\hat{\imath} + yz\hat{\jmath} + xyz\hat{\mathbf{k}}$ through the sides (not the ends) of the cylinder of radius a and lying between $0 \le z \le h$.

Mastery Part

8. (4 p) Parameterize the integral curve of $\bar{\mathbf{V}} = -y\hat{\imath} + x\hat{\jmath} + c\hat{\mathbf{k}}$ starting at $(1,0,0)$ by its arclength.

9. (4 p) Consider the directional derivative defined by

$$D_{\bar{\mathbf{u}}} f(a,b) := \frac{d}{dt}\Big|_{t=0} f(a + tu_1, b + tu_2),$$

at a point (a,b) where $f(x,y)$ is differentiable and where $\bar{\nabla} f(a,b) \neq 0$. What are the maximum and minimum values of $D_{\bar{\mathbf{u}}} f(a,b)$? Does $D_{\bar{\mathbf{u}}} f(a,b)$ vanish? If so, in what directions? Support your answers.

10. (4 p) Use the divergence theorem to find the flux of $\bar{\mathbf{V}} = (x + xz)\hat{\imath} + (x + yz)\hat{\jmath} - (2x + z^2)\hat{\mathbf{k}}$ out of the first octant of the sphere $x^2 + y^2 + z^2 = a^2$.

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1/ a) $f(x,y) = x^3 - 10x + y^2 - 6$

$f_1 = 3x^2 - 10 \quad , \quad f_2 = 2y \Rightarrow \boxed{\bar{\nabla} f = (3x^2 - 10)\hat{i} + 2y\hat{j}}$

b) $\bar{N} = -f_1 \hat{i} - f_2 \hat{j} + \hat{k}$

at $(x,y) = (2,1) \qquad \bar{N} = -2\hat{i} - 2\hat{j} + \hat{k}$

$|\bar{N}| = 3 \Rightarrow \boxed{\hat{N}\Big|_{P_0} = -\tfrac{2}{3}\hat{i} - \tfrac{2}{3}\hat{j} + \tfrac{1}{3}\hat{k}}$

c) $\bar{r}(t) = P_0 + \hat{N} t \quad , \quad f(2,1) = 8 - 20 + 1 - 6 = -17$

$$\boxed{\therefore \ \bar{r}(t) = (2 - \tfrac{2}{3}t)\hat{i} + (1 - \tfrac{2}{3}t)\hat{j} + (-17 + \tfrac{1}{3}t)\hat{k}}$$

2/ $P_2(f,(0,0)) = f(0,0) + \bar{h}\cdot\bar{\nabla} f\Big|_{(0,0)} + \frac{1}{2}(\bar{h}\cdot\bar{\nabla})^2 f\Big|_{(0,0)}$

where $\bar{h} = (x,y)$.

For $f(x,y) = e^{3y-x} \qquad f(0,0) = 1$

$\bar{\nabla} f = -e^{3y-x}\hat{i} + 3e^{3y-x}\hat{j} \quad , \quad \bar{\nabla} f\Big|_{(0,0)} = -\hat{i} + 3\hat{j}$

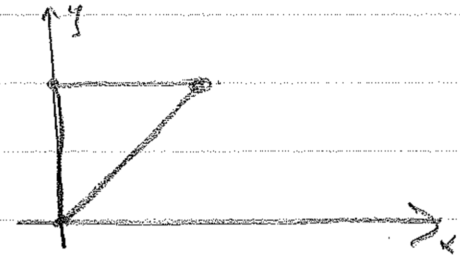
$(\bar{h}\cdot\bar{\nabla})f = \bar{h}^T \mathrm{Hess}(f)\, \bar{h} \quad , \quad \mathrm{Hess}(f)\Big|_{(0,0)} = \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix}\Big|_{(0,0)} = \begin{pmatrix} 1 & -3 \\ -3 & 9 \end{pmatrix}$

$= (3y - x)^2$

So,

$P_2(e^{3y-x},(0,0)) = 1 + 3y - x + \frac{1}{2}(3y - x)^2$

3/ $f(x,y) = 2xy - x - \frac{1}{2}y^2$



Interior critical points:

$\bar{\nabla} f = (2y-1)\hat{i} + (2x-y)\hat{j}$

$\bar{\nabla} f = 0 \Rightarrow y = 1/2$ and $2x = y \Rightarrow x = 1/4$

Further, $f(1/4, 1/2) = 1/4 - 1/4 - 1/2(1/4) = -1/8$

Boundary Points.

Corners: $(0,0)$, $f(0,0) = 0$

$(0,1)$, $f(0,1) = -1/2$

$(1,1)$, $f(1,1) = 2 - 1 - 1/2 = 1/2$

Lines: $x = 0$ $\quad f(0,y) = -\frac{1}{2}y^2$ dec. func. with min. $-1/2$

$y = 1$ $\quad f(x,1) = 2x - x - 1/2$ inc. func. min $-1/2$, max $1/2$

$x = y$ $\quad f(x,x) = \frac{3}{2}x^2 - x = x(\frac{3}{2}x - 1)$

g(x) := f(x,x)= 1,5x^2 - x
g'(x) = 3x - 1
g'(x) = 0 omm x=1/3
x=1/3 är en kritisk punkt

~~crit. point at $x = 2/3$~~

$\cancel{f(2/3, 2/3) = 2\frac{4}{9} - \frac{2}{3} - \frac{1}{2}(\frac{4}{9}) = 0}$

Detta är fel!

All possible extreme points on the triangle are in

$\{(1/4, 1/2), (0,0), (0,1), (1,1), (2/3, 2/3)\}$

4/ $$\int_{z=0}^{1}\int_{y=0}^{1-z}\int_{x=0}^{1-y-z} z\,dx\,dy\,dz$$

$$= \int_{z=0}^{1}\int_{y=0}^{1-z}\left(z(1-z)-zy\right)dy\,dz$$

$$= \int_{z=0}^{1}\left[z(1-z)^2-\frac{1}{2}z(1-z)^2\right]dz$$

$$= \frac{1}{2}\int_{0}^{1}\left(z-2z^2+z^3\right)dz$$

$$= \frac{1}{2}\left(\frac{1}{2}-\frac{2}{3}+\frac{1}{4}\right) = \boxed{1/24}$$

5/ $$\operatorname{div}\bar{V} = e^{yz}\hat{i} + e^{xz}\hat{j} + e^{xy}\hat{k}$$

$$\operatorname{curl}\bar{V} = \left(\frac{\partial V^3}{\partial y}-\frac{\partial V^2}{\partial z}\right)\hat{i} - \left(\frac{\partial V^3}{\partial x}-\frac{\partial V^1}{\partial z}\right)\hat{j} + \left(\frac{\partial V^2}{\partial x}-\frac{\partial V^1}{\partial y}\right)\hat{k}$$

$$= \left(xze^{xy}-xye^{xz}\right)\hat{i} - \left(yze^{xy}-xye^{yz}\right)\hat{j}$$

$$+ \left(yze^{xz}-xze^{yz}\right)\hat{k}$$

$$= xe^{x}\left(ze^{y}-ye^{z}\right)\hat{i} + ye^{y}\left(xe^{z}-ze^{x}\right)\hat{j}$$

$$+ ze^{z}\left(ye^{x}-xe^{y}\right)\hat{k}$$

6/ By Green's Theorem

$$\text{Area} = \iint\left(\frac{\partial F^2}{\partial x} - \frac{\partial F^1}{\partial y}\right)dA \;=\; -\oint_C \bar{F}\cdot d\bar{r}$$

for $\bar{F} = -y\hat{i}$; minus b/c clockwise.

Now $\bar{v}(t) = 6\cos(2t)\hat{i} + \left(-2\sin(2t) - 4\sin(4t)\right)\hat{j}$

$\bar{F}(\bar{r}(t)) = -\left(\cos(2t) + \cos(4t)\right)\hat{i}$

$$\therefore -\oint \bar{F}\cdot d\bar{r} = 6\int_{t=0}^{\pi}\left[\cos^2(2t) + \cos(2t)\cos(4t)\right]dt$$

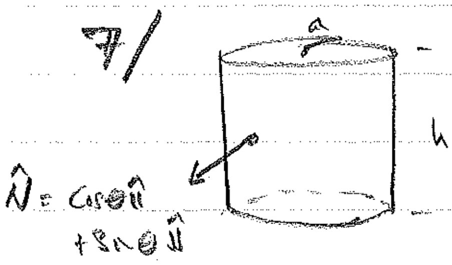
$$= 3\int_{u=0}^{2\pi}\cos^2(u)\,du + \frac{3}{2}\int_0^{2\pi}\left(\cos(u) + \cos(3u)\right)du$$

// by $u = 2t$, and $\cos(x)\cos(y) = \frac{1}{2}\left(\cos(x+y) + \cos(x-y)\right)$ //

$$= 3\left[\frac{u}{2} + \frac{1}{4}\sin(2u)\right]_0^{2\pi} + 0$$

$$\boxed{= 3\pi}$$

7/



$$\Phi = \iint_{S'}\left(\bar{V}\cdot\hat{N}\right)\Big|_{S'} dS$$

$$= \iint\left[(xz + 2)\cos\theta + yz\sin\theta\right]\Big|_{S'} dS$$

$$= \int_{z=0}^{h}\int_{\theta=0}^{2\pi}\left(az\cos^2\theta + 2\cos\theta + az\sin^2\theta\right)a\,d\theta\,dz$$

$$= 2\pi a^2\int_0^h z\,dz + 2ah\,\cancel{\int_0^{2\pi}\cos\theta\,d\theta} \to 0$$

$$\boxed{= \pi a^2 h^2}$$

8/ The integral curve of $\bar{V} = -y\hat{i} + x\hat{j} + c\hat{k}$, $c > 0$ satisfies

$$\frac{dx}{dt} = -y \quad , \quad x(t=0) = 1$$

$$\frac{dy}{dt} = x \quad , \quad y(t=0) = 0$$

$$\frac{dz}{dt} = c \quad , \quad z(t=0) = 0$$

$\Rightarrow$ $x(t) = \cos(t)$, $y(t) = \sin(t)$, $z(t) = ct$

so

$$\bar{r}(t) = \cos(t)\,\hat{i} + \sin(t)\,\hat{j} + ct\,\hat{k}$$

To find arc length we compute

$$s = \int_0^t v(\tilde{t})\,d\tilde{t} \quad , \quad v(t) = \sqrt{1+c^2} \text{ obtained from } \bar{r}(t).$$

$$= t\sqrt{1+c^2}$$

Thus $t = s/\sqrt{1+c^2}$

and we write the curve in terms of the arc length

$$\boxed{\bar{r}(s) = \cos\left(\frac{s}{\sqrt{1+c^2}}\right)\hat{i} + \sin\left(\frac{s}{\sqrt{1+c^2}}\right)\hat{j} + \frac{c}{\sqrt{1+c^2}}\,s\,\hat{k}}$$

9/ By the chain rule

$$D_{\bar{u}} f(a,b) = \frac{d}{dt}\Big|_{t=0} f(a+tu_1, b+tu_2)$$

$$= \left(u_1 f_1(a+tu_1, b+tu_2) + u_2 \cdot f_2(a+tu_1, b+tu_2)\right)\Big|_{t=0}$$

$$= u_1 f_1(a,b) + u_2 f_2(a,b)$$

$$= \bar{u} \cdot \bar{\nabla} f(a,b)$$

Now $\bar{u} \cdot \bar{\nabla} f(a,b) = |\bar{\nabla} f| \cos\theta$, where

θ is angle between $\bar{u}$ and $\bar{\nabla} f$.

and where we used that $\bar{u}$ is unit.

Since $\cos(\theta) \in [-1, 1]$ $D_{\bar{u}} f(a,b)$ has a minimum of $-|\bar{\nabla} f(a,b)|$ and maximum of $|\bar{\nabla} f(a,b)|$ in $-\bar{\nabla} f(a,b)$ and $\bar{\nabla} f(a,b)$ directions respectively.

Further $D_{\bar{u}} f(a,b) = 0$ when $\theta = \pi/2$, that is when $\bar{u} \perp \bar{\nabla} f(a,b)$. This is the directions tangent to the level curves of f. To show this let $x(t), y(t)$ parameterize the level curve of f through (a,b) with $x(0) = a$, $y(0) = b$. Thus $f(a,b) = f(x(t), y(t))$

$$\Rightarrow \quad 0 = \frac{d}{dt}\Big|_{t=0} f(x(t), y(t)) = \bar{v} \cdot \bar{\nabla} f(a,b)$$

by the chain rule where $\bar{v}$ is the velocity, tangent to the level curve at (a,b). This proves the above claim.

10/ Div. Th'm $\iiint \operatorname{div} \bar{F}\, dV = \oiint_{S} \bar{F}\cdot\hat{N}\, dS$

$$\Phi \equiv \iint_{S'} \bar{F}\cdot\hat{N}\, dS = \iiint \operatorname{div}\bar{F}\, dV - \sum_{i=1}^{3}\iint_{P_i} \bar{F}\cdot\hat{N}_i\, dS$$

↑ sum over coord. planes

$$\bar{V} = (x+xz)\hat{i} + (x+yz)\hat{j} - (2x+z^2)\hat{k}$$

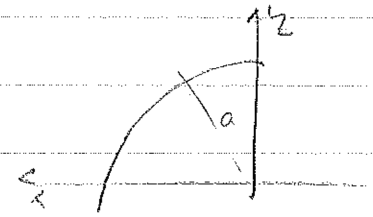
$$\operatorname{div}\bar{V} = 1+z+z-2z = 1$$

$$\Rightarrow \iiint \operatorname{div}\bar{V}\, dV = \tfrac{1}{8}\,\mathrm{Vol}\,(\text{sphere}) = \tfrac{1}{8}\,\tfrac{4}{3}\pi a^3 = \frac{\pi a^3}{6}$$

P_x: $\hat{N} = -\hat{i}$ $\quad \bar{V}\cdot\hat{N}\big|_{P_x} = -(x+xz)\big|_{x=0} = 0$

P_y: $\hat{N} = -\hat{j}$; $\bar{V}\cdot\hat{N}\big|_{P_y} = (-x-yz)\big|_{P_y} = -x$

$$-\iint_{P_y} x\, dx\, dz = -\int_{\theta=0}^{\pi/2}\int_{r=0}^{a} r\cos\theta\, r\, dr\, d\theta$$



$$= -\tfrac{1}{3}a^3 \cdot 1$$

P_z: $\hat{N} = -\hat{k}$; $\bar{V}\cdot\hat{N}\big|_{P_z} = +2x$; $2\iint_{P_z} x\, dx\, dy = \tfrac{2}{3}a^3$

so, $\Phi = \frac{\pi}{6}a^3 - \frac{1}{3}a^3 = \frac{a^3}{3}\left(\frac{\pi}{2} - 1\right)$