MVE041 - Omtentan 2017-08-25

(1) $f(x,y,z) = e^z + (x+y)z - x^3 + y$

$\nabla f = (z - 3x^2,\ z+1,\ e^z + x + y)$

(a) $\nabla f(1,2,0)$ är normalvektor till tan-planet

$\nabla f(1,2,0) = (0 - 3\cdot 1^2,\ 0+1,\ e^0 + 1 + 2) = (-3, 1, 4)$

Planets ekvation blir $-3x + y + 4z = D$

Punkten $(1,2,0)$ ligger i planet $\Rightarrow -3\cdot 1 + 2 + 4\cdot 0 = D$

$-1 = D$

Tangentplanets ekvation är $\underline{\underline{-3x + y + 4z = -1}}$

(b) f avtar som snabbast i riktningen $\vec{v} = -\nabla f(1,2,0)$

$$D_{\vec{v}} f = \nabla f(1,2,0) \bullet \frac{\vec{v}}{\|\vec{v}\|} = \frac{\nabla f(1,2,0) \bullet (-\nabla f(1,2,0))}{\|-\nabla f(1,2,0)\|} =$$

$$= -\|\nabla f(1,2,0)\| = -\sqrt{(-3)^2 + 1^2 + 4^2} = \underline{\underline{-\sqrt{26}}}$$

(2) $f(x,y) = (y^2 - 2x)\, e^{x-2y}$

$\nabla f(x,y) = \left(-2x e^{x-2y} + (y^2-2x)e^{x-2y},\ 2y e^{x-2y} + (y^2-2x)e^{x-2y}(-2)\right)$

$\nabla f(x,y) = e^{x-2y} \cdot (y^2 - 4x,\ 4x + 2y - 2y^2)$

(a) $\nabla f = \vec{0} \Leftrightarrow \begin{cases} y^2 - 4x = 0 \Rightarrow 4x = y^2 \\ 4x + 2y - 2y^2 = 0 \end{cases} \Rightarrow y^2 + 2y - 2y^2 = 0$

$2y - y^2 = 0$

$y \cdot (2 - y) = 0$

(i) $y = 0 \Rightarrow x = \frac{0^2}{4} = 0$

(ii) $y = 2 \Rightarrow x = \frac{2^2}{4} = 1$

För att bestämma karaktär, så behöver andra-derivatan beräknas:

$$\nabla f = \left(e^{x-2y}(y^2-4x),\ e^{x-2y}(4x+2y-2y^2)\right)$$

$$\frac{\partial^2 f}{\partial x^2} = e^{x-2y}(y^2-4x-4)$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = e^{x-2y}\left(-2(y^2-4x)+2y\right) = e^{x-2y}(8x+2y-2y^2)$$

$$\frac{\partial^2 f}{\partial y^2} = e^{x-2y}\left(-2(4x+2y-2y^2)+(2-4y)\right) =$$

$$= e^{x-2y}(4y^2-8y-8x+2)$$

(i) $(x,y)=(0,0)$:

$$\mathcal{H}_f(0,0) = \begin{pmatrix} e^0\cdot(-4) & e^0\cdot 0 \\ e^0\cdot 0 & e^0\cdot 2 \end{pmatrix} = \begin{pmatrix} -4 & 0 \\ 0 & 2 \end{pmatrix}$$

$\det \mathcal{H}_f(0,0) = -8 < 0 \Rightarrow$ Indefinit

$\Downarrow$

Den stationära punkten $(0,0)$ är en sadelpunkt.

(ii) $(x,y)=(1,2)$:

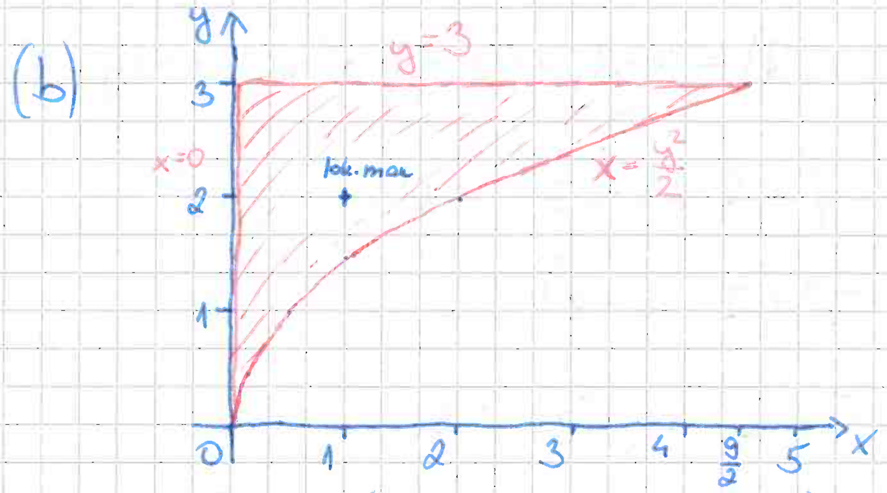
$$\mathcal{H}_f(1,2) = \begin{pmatrix} e^{-3}(-4) & e^{-3}4 \\ e^{-3}4 & e^{-3}(-6) \end{pmatrix} = e^{-3}\begin{pmatrix} -4 & 4 \\ 4 & -6 \end{pmatrix}$$

$\det \mathcal{H}_f(1,2) = e^{-9}\cdot(24-16) = e^{-9}\cdot 8 > 0$

& diagonalelementen är negativa $\Longrightarrow$ neg. definit

$\therefore$ Den stationära punkten $(1,2)$ är lok. max.

(b)



Inuti D: lok. max. i pkten $(1,2)$: $f(1,2) = e^{-3} \cdot 2$

Randen: (i) $x=0,\ 0 \le y \le 3$:

$g(y) = f(0,y) = y^2 \cdot e^{-2y}$ $\qquad g(0)=0,\quad g(3) = 9e^{-6}$

$g'(y) = 2y\,e^{-2y} - 2y^2 e^{-2y} = 2y\,e^{-2y}(1-y)$

$g'(y) = 0 \iff y=0$ el. $y=1$

$g(0) = f(0,0) = 0 \qquad g(1) = f(0,1) = e^{-2}$

(ii) $y=3,\quad 0 \le x \le 4{,}5$:

$g(x) = f(x,3) = (9-2x)e^{x-6}$

$g'(x) = -2e^{x-6} + (9-2x)e^{x-6} = (7-2x)e^{x-6}$

$g'(x) = 0 \iff x = 3{,}5$

$g(0) = 9e^{-6} \qquad g(3{,}5) = 2e^{-2{,}5} \qquad g(4{,}5) = 0$

(iii) $x = y^2/2,\quad 0 \le y \le 3$

$g(y) = f\left(\frac{y^2}{2}, y\right) = 0$

I området D är f icke-negativt $\Rightarrow$ MIN värde $=0$

Kandidater för max: $\frac{2}{e^3},\ \frac{9}{e^6},\ \frac{1}{e^2},\ \frac{2}{e^{2,5}}$

- $9 = 3^2 < e^3 \Rightarrow \frac{9}{e^6} < \frac{e^3}{e^6} = \frac{1}{e^3} < \frac{2}{e^3}$ $\Rightarrow \frac{9}{e^6}$ ÄR EJ MAX
- $2 < e \Rightarrow \frac{2}{e^3} < \frac{e}{e^3} = \frac{1}{e^2}$ $\Rightarrow$ $2/e^3$ är ej max
- $\frac{1}{e^2} = \frac{\sqrt{e}}{e^{2,5}} < \frac{\sqrt{3}}{e^{2,5}} < \frac{1{,}8}{e^{2,5}} < \frac{2}{e^{2,5}} \Rightarrow$ $2/e^{2,5}$ är MAX

(3) $L(x,y,z,\lambda) = x-2y+5z+\lambda(x^2+2y^2+xz+yz+z^2-1)$

$\frac{\partial L}{\partial x} = 1+\lambda(2x+z) = 0$ (I)

$\frac{\partial L}{\partial y} = -2+\lambda(4y+z) = 0$ (II)

$\frac{\partial L}{\partial z} = 5+\lambda(x+y+2z) = 0$ (III)

$\frac{\partial L}{\partial \lambda} = x^2+2y^2+xz+yz+z^2-1=0$ (IV)

(I) $\lambda(2x+z) = -1$ $(\Rightarrow \lambda\neq 0,\ 2x+z\neq 0)$

(II) $\lambda(4y+z) = 2$ $(\Rightarrow 4y+z\neq 0,\ \lambda\neq 0)$

(III) $\lambda(x+y+2z) = -5$ $(\Rightarrow \lambda\neq 0,\ x+y+2z\neq 0)$

(I) $\Rightarrow$ $2x+z=\frac{-1}{\lambda}$ (II) $\Rightarrow$ $4y+z=\frac{2}{\lambda}$ (III) $x+y+2z=\frac{-5}{\lambda}$

(I) & (II) $\Rightarrow$ $4y+z=-2\cdot(2x+z)$ $\Leftrightarrow$ $4x+4y+3z=0$ (V)

(I) & (III) $\Rightarrow$ $x+y+2z=5(2x+z)$ $\Leftrightarrow$ $9x-y+3z=0$ (VI)

(VI) $-$ (V): $5x-5y=0$ $\Rightarrow$ $\boxed{x=y}$ (VII)

(VII) $\rightarrow$ (V): $-8x=3z$ $\Rightarrow$ $\boxed{z=\frac{-8}{3}x}$ (VIII)

(VII) & (VIII) $\rightarrow$ (IV):

$$x^2+2x^2-\frac{8}{3}x^2-\frac{8}{3}x^2+\frac{64}{9}x^2-1=0$$

$$3x^2+\frac{64-48}{9}x^2=1 \Leftrightarrow \frac{43}{9}x^2=1$$

$$x=\pm\frac{3}{\sqrt{43}}$$

- $x=\frac{3}{\sqrt{43}}=y$, $z=\frac{-8}{\sqrt{43}}$ $f\left(\frac{3}{\sqrt{43}},\frac{3}{\sqrt{43}},\frac{-8}{\sqrt{43}}\right)=-\sqrt{43}$
- $x=\frac{-3}{\sqrt{43}}=y$, $z=\frac{8}{\sqrt{43}}$ $f\left(\frac{-3}{\sqrt{43}},\frac{-3}{\sqrt{43}},\frac{8}{\sqrt{43}}\right)=\sqrt{43}$

Lagrange multiplikatorer funkar ej ifall

$\nabla g = 0$, d.v.s., ifall $\left.\begin{array}{r} 2x+z=0 \\ 4y+z=0 \\ x+y+2z=0 \end{array}\right\} \Leftrightarrow x=y=z=0$

Punkten $(0,0,0)$ ligger ej på ytan $g(x,y,z)=0$.

$\Rightarrow$ Lagrange har hittat samtliga extremvärden.

Är ytan begränsad?

$$x^2+2y^2+xz+yz+z^2=1 \quad \Leftrightarrow \quad /\text{kvadratkomplettering}/$$

$$\Leftrightarrow \left(x+\tfrac{1}{2}z\right)^2-\frac{z^2}{4}+2\cdot\left(y+\frac{z}{4}\right)^2-\frac{z^2}{16}+z^2=1$$

$$\Leftrightarrow \left(x+\frac{z}{2}\right)^2+2\left(y+\frac{z}{4}\right)^2+\frac{11}{16}z^2=1$$

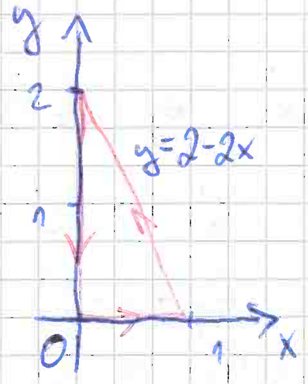
Ytan är en ellipsoid $\Rightarrow$ den är sluten,
sammanhängande
o begränsad.

✓

Värdemängden är intervallet mellan
min- o maxvärdena som hittats m.h.a.
Lagrange

$$V_f=\left[-\sqrt{43}\,;\,\sqrt{43}\right]$$

(4)



(a) (i) $\xrightarrow{y=0}$ $\int_{\rightarrow} \vec{F}\cdot d\vec{r} = \int_0^1 y^2\,dx =$

$= \int_0^1 0\,dx = 0$

(ii) $\nwarrow$ $y=2-2x$ $\quad \int_{\nwarrow} \vec{F}\cdot d\vec{r} = \int_1^0 \begin{pmatrix} y^2 \\ -x^2 \end{pmatrix}\cdot\begin{pmatrix} 1 \\ -2 \end{pmatrix} dx$

$= \int_0^1 \begin{pmatrix} (2-2x)^2 \\ -x^2 \end{pmatrix}\cdot\begin{pmatrix} -1 \\ 2 \end{pmatrix} dx$

$= \int_0^1 -4+8x-4x^2-2x^2\,dx$

$= \left[-4x+4x^2-2x^3\right]_0^1 = -2$

(iii) $\downarrow$ $x=0$ $\quad \int_{\downarrow} \vec{F}\cdot d\vec{r} = \int_0^2 \begin{pmatrix} (2-t)^2 \\ 0 \end{pmatrix}\cdot\begin{pmatrix} 0 \\ -1 \end{pmatrix} dt = \int_0^2 0 = 0$

$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 2-t \end{pmatrix}$

$t\in[0,2]$

$\oint_\Delta \vec{F}\cdot d\vec{r} = 0-2+0 = \underline{\underline{-2}}$

(b) $\oint_\Delta \vec{F}\cdot d\vec{r} = \iint_\Delta \left(\frac{\partial(-x^2)}{\partial x} - \frac{\partial y^2}{\partial y}\right) dx\,dy =$

$= \iint_\Delta (-2x-2y)\,dx\,dy$

$= \int_0^1 \int_0^{2-2x} (-2x-2y)\,dy\,dx =$

$= -2\int_0^1 \left[xy+\frac{y^2}{2}\right]_0^{2-2x} dx = -2\cdot\int_0^1 2x-2x^2+2(1-2x+x^2)\,dx$

$= -2\cdot\int_0^1 \ldots x^2\,dx = -2\cdot\left[2x-x^2 \ldots\right]_0^1$

$(2-2x) \ldots^2$

$= -2\cdot \ldots$

(5) $m = \iint_{\varphi} \rho(x,y,z)\, dS$

sfäriska koordinater:

$$x = 2\cos\theta \sin\varphi$$
$$y = 2\sin\theta \sin\varphi$$
$$z = 2\cos\varphi$$

villkoret $\sqrt{2} \le z \le 2 \iff \sqrt{2} \le 2\cos\varphi \le 2$

$$\Updownarrow$$
$$\frac{\sqrt{2}}{2} \le \cos\varphi \le 1$$
$$\Updownarrow$$

$\boxed{\theta \in [-\pi, \pi]}$ $\boxed{\frac{\pi}{4} \ge \varphi \ge 0}$

$$\frac{d\vec{r}}{\partial\theta} = \begin{pmatrix} -2\,s\theta\, s\varphi \\ 2\,c\theta\, s\varphi \\ 0 \end{pmatrix} \qquad \frac{d\vec{r}}{\partial\varphi} = \begin{pmatrix} 2\,c\theta\, c\varphi \\ 2\,s\theta\, c\varphi \\ -2\,s\varphi \end{pmatrix}$$

$$\frac{d\vec{r}}{d\theta} \times \frac{d\vec{r}}{d\varphi} = \begin{pmatrix} -4c\theta\, s^2\varphi \\ -4\,s\theta\, s^2\varphi \\ -4\, s\varphi c\varphi \end{pmatrix}$$

$$\left\| \frac{d\vec{r}}{d\theta} \times \frac{d\vec{r}}{d\varphi} \right\| = 4\cdot\sqrt{c^2\theta s^4\varphi + s^2\theta s^4\varphi + s^2\varphi c^2\varphi}$$

$$= 4\cdot\sqrt{s^4\varphi + s^2\varphi c^2\varphi}$$

$$= 4\, s\varphi \sqrt{s^2\varphi + c^2\varphi} = 4\,s\varphi$$

$$m = \iint_{\varphi} z\, dS = \int_{-\pi}^{\pi}\int_0^{\pi/4} \underbrace{4\cos\varphi}_{=z} \cdot \underbrace{4\sin\varphi\, d\varphi\, d\theta}_{=dS} =$$

$$= \int_{-\pi}^{\pi} d\theta \int_0^{\pi/4} 8\sin 2\varphi\, d\varphi = 2\pi \cdot \left[-4\cos 2\varphi\right]_0^{\pi/4}$$

$$= 2\pi \cdot 4 = \underline{8\pi}$$

(6)

$$\iint_{\partial K} \vec{F} \bullet \hat{N} \, dS = \iiint_K \operatorname{div} \vec{F} \, dV$$

$$\operatorname{div} \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = 0 + z + 0$$

Tetraederns sidor: $x=0$, $y=0$, $z=0$,

$$x+2y+4z=12$$

x i innersta $\int$: $0 \le x \le 12-2y-4z$

y i mellersta $\int$: $0 \le y$ & $0 \le 12-2y-4z$

$$0 \le y \le 6-2z$$

z i yttersta $\int$: $0 \le z$ & $0 \le 6-2z$

$$0 \le z \le 3$$

$$\iiint_K \operatorname{div} \vec{F} \, dV = \int_0^3 \int_0^{6-2z} \int_0^{12-2y-4z} z \, dx \, dy \, dz$$

$$= \int_0^3 z \int_0^{6-2z} 12-2y-4z \, dy \, dz$$

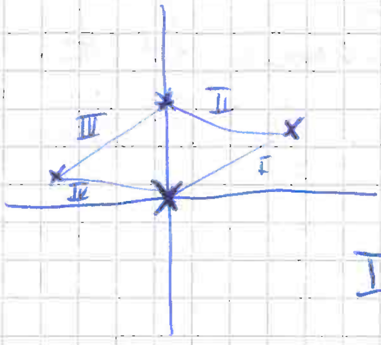
$$= \int_0^3 z \left[12y - y^2 - 4zy \right]_0^{6-2z} dz$$

$$= \int_0^3 z \left(72 - 24z - 36 + 24z - 4z^2 - 24z + 8z^2 \right) dz$$

$$= \int_0^3 36z - 24z^2 + 4z^3 \, dz$$

$$= \left[18z^2 - 8z^3 + z^4 \right]_0^3 = 162 - 216 + 81 = \underline{\underline{27}}$$

(7)



$\mathrm{I}: \quad y = \frac{2}{3}x \qquad\qquad \mathrm{III}: \quad y = \frac{2}{3}x + 3$

$\mathrm{II}: \quad y = 3 - \frac{1}{3}x \qquad\qquad \mathrm{IV}: \quad y = -\frac{1}{3}x$

$\mathrm{I} \,\&\, \mathrm{III}: \qquad 0 \overset{(\mathrm{I})}{\leq} 3y - 2x \overset{(\mathrm{III})}{\leq} 9$

$\mathrm{II} \,\&\, \mathrm{IV} \qquad 0 \overset{(\mathrm{IV})}{\leq} 3y + x \overset{(\mathrm{II})}{\leq} 9$

variabelbyte: $\quad u = 3y - 2x \qquad \det \frac{\partial(u,v)}{\partial(x,y)} = \det \begin{pmatrix} -2 & 3 \\ 1 & 3 \end{pmatrix}$

$\quad v = 3y + x$

$\Downarrow \qquad\qquad\qquad = \cancel{-12}\ -9$

$u + 2v = 9y$

$\left|\det \frac{\partial(x,y)}{\partial(u,v)}\right| = \cancel{\tfrac{1}{12}}\ \frac{1}{9}$

$$\iint_D y \sin(3y-2x)\,dx\,dy = \int_0^9 \int_0^9 \frac{u+2v}{9} \sin(u) \frac{1}{9}\,dv\,du$$

$$= \frac{1}{81} \int_0^9 \left[(uv + v^2)\sin u\right]_{v=0}^{9} du =$$

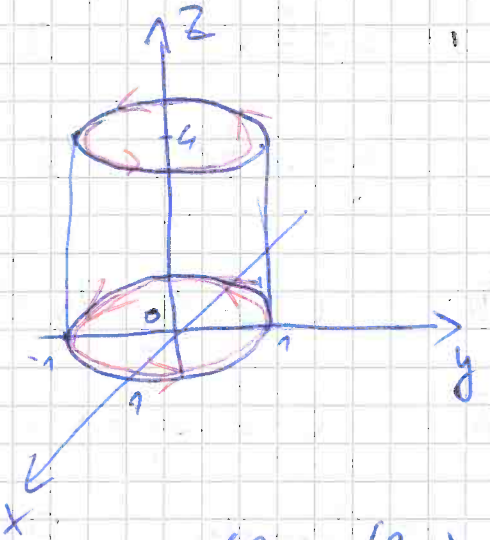
$$= \frac{1}{9} \int_0^9 (u+9) \sin u \, du = \left(\begin{array}{ll} \text{partiell integration} & \\ f = u+9 & g' = \sin u \\ f' = 1 & g = -\cos u \end{array} \right)$$

$$= \frac{1}{9}\left(\left[(u+9)(-\cos u)\right]_0^9 + \int_0^9 \cos u \, du \right)$$

$$= \frac{1}{9}\left(-18\cos 9 + 9\cos 0 + \left[\sin u\right]_0^9 \right) =$$

$$= -2\cos 9 + 9 + \frac{1}{9}\sin 9$$

(8)



Randen består av $\underline{\underline{\text{två}}}$ cirklar!

$$\text{curl } \vec{F} = \begin{pmatrix} 2xy-(2y-1)x \\ y^2+y-y^2 \\ (2y-1)z-(2y+1)z \end{pmatrix} = \begin{pmatrix} x \\ y \\ -2z \end{pmatrix}$$

Undre cirkeln

$$\oint_{C_u} \vec{F}\bullet d\vec{r} = \iint_{D_u} \text{curl } \vec{F}\bullet \underbrace{\hat{N}}_{=\begin{pmatrix}0\\0\\1\end{pmatrix}} dS = \iint_{D_u} (-2z)\, dS = 0$$

↑ cirkelskivan $x^2+y^2\le 1,\ z=0$

Övre cirkeln

$$\oint_{C_ö} \vec{F}\bullet d\vec{r} = \iint_{D_ö} \text{curl } \vec{F}\bullet \overset{\begin{pmatrix}0\\0\\1\end{pmatrix}}{\hat{N}}\, dS = \iint_{D_ö} (-2z)\, dS = (*)$$

↑ cirkelskivan $x^2+y^2\le 1,\ z=4$

$$(*) = \iint_{x^2+y^2\le 1} -8\, dx\, dy = -8\cdot\pi\cdot 1^2 = -8\pi$$

$$\oint_{C} \vec{F}\bullet d\vec{r} = \oint_{C_u} \vec{F}\bullet d\vec{r} + \oint_{C_ö} \vec{F}\bullet d\vec{r} = 0 - 8\pi = \underline{\underline{-8\pi}}$$