Övningstenta MVE04T. vt2017

1 (a) $\nabla f = (z^2 + y^2 e^{xy},\ e^{xy} + xye^{xy},\ 2xz)$

f avtar som snabbast i riktningen $-\nabla f(\mathbf{x}_0)$ i punkten $\mathbf{x}_0$

$\Rightarrow \ -\nabla f(1,0,-1) = -(1,1,-2) = (-1,-1,2)$

(b) $\mathbf{u} = (-1,2,1) \Rightarrow |\mathbf{u}| = \sqrt{1+4+1} = \sqrt{6}$

$\Rightarrow \hat{\mathbf{u}} = \frac{1}{\sqrt{6}}(-1,2,1)$

$D_{\hat{\mathbf{u}}} f(1,0,-1) = \hat{\mathbf{u}} \cdot \nabla f(1,0,-1) =$

$= \frac{1}{\sqrt{6}}(-1,2,1)\cdot(1,1,-2) = -\frac{1}{\sqrt{6}}$

(c) $\mathbf{N} = \nabla f(1,0,-1) = (1,1,-2)$

$\Rightarrow \quad x + y - 2z = D$. $(1,0,-1)$ ligger i planet

$\Rightarrow \ D = 1 + 0 + 2 = 3$

$\therefore \ x + y - 2z = 3$

2. $\nabla f = \left(2xy - y^3 + 2x \,,\; x^2 - 3xy^2 \right)$

$$\nabla f = 0 \iff \begin{cases} 2xy - y^3 + 2x = 0 \\ x^2 - 3xy^2 = 0 \end{cases}$$

$$x^2 - 3xy^2 = 0 \iff x(x - 3y^2) = 0$$

<u>Fall I</u>: $x = 0 \Rightarrow -y^3 = 0 \iff y = 0$

$\Rightarrow (x_1, y_1) = (0,0)$ ← kritisk punkt

<u>Fall II</u>: $x \neq 0 \Rightarrow x = 3y^2 \Rightarrow$

$\Rightarrow 2 \cdot 3y^2 \cdot y - y^3 + 2 \cdot 3y^2 = 0 \iff 5y^3 + 6y^2 = 0$

$\iff y^2(5y + 6) = 0 \Rightarrow y_1 = 0$ ← Fall I

$y_2 = -\frac{6}{5} \Rightarrow x_2 = 3 \cdot \left(-\frac{6}{5}\right)^2 = \frac{108}{25}$

$\therefore (x_2, y_2) = \left(\frac{108}{25}, -\frac{6}{5} \right)$ ← kritisk punkt

$$\mathcal{H}(x,y) = \begin{pmatrix} 2y + 2 & 2x - 3y^2 \\ 2x - 3y^2 & -6xy \end{pmatrix}$$

$\mathcal{H}(0,0) = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \Rightarrow \begin{array}{l} \lambda_1 = 2 \\ \lambda_2 = 0 \end{array}$ ← Ingen slutsats kan dras!

$f(\varepsilon, 0) = \varepsilon^2 > 0$ för små $\varepsilon > 0$

$f(\varepsilon^4, \varepsilon) = \varepsilon^9 - \varepsilon^7 + \varepsilon^8 = \varepsilon^7(\varepsilon^2 + \varepsilon - 1) < 0$ för små $\varepsilon > 0$

$\therefore (0,0)$ sadelpunkt

$$\mathcal{H}\left(\frac{108}{25}, -\frac{6}{5}\right) = \begin{pmatrix} -\frac{12}{5}+2 & \frac{2\cdot 108}{25} - \frac{3\cdot 36}{25} \\ \frac{2\cdot 108}{25} - 3\cdot\frac{36}{25} & -6\cdot\frac{108}{25}\cdot\left(-\frac{6}{5}\right) \end{pmatrix} =$$

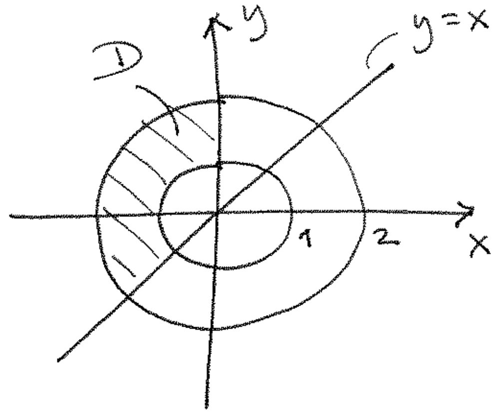
$$= \begin{pmatrix} -2/5 & 108/25 \\ 108/25 & \frac{36}{5}\cdot\frac{108}{25} \end{pmatrix}$$

$$\Rightarrow \det \mathcal{H}\left(\frac{108}{25}, -\frac{6}{5}\right) = -\frac{72}{25}\cdot\frac{108}{25} - \frac{108}{25}\cdot\frac{108}{25} < 0$$

$\therefore \left(\frac{108}{25}, -\frac{6}{5}\right)$ sadelpunkt

3. Parametrisering av S:

$$\mathbb{r}(x,y) = \left(x, y, \sqrt{4-x^2-y^2}\right), \quad (x,y) \in D$$



$$\frac{\partial \mathbb{r}}{\partial x} = \left(1, 0, -\frac{x}{\sqrt{4-x^2-y^2}}\right)$$

$$\frac{\partial \mathbb{r}}{\partial y} = \left(0, 1, -\frac{y}{\sqrt{4-x^2-y^2}}\right)$$

$$\Rightarrow \left|\frac{\partial \mathbb{r}}{\partial x} \times \frac{\partial \mathbb{r}}{\partial y}\right| = \left\| \begin{matrix} \hat{\mathbb{i}} & \hat{\mathbb{j}} & \hat{\mathbb{k}} \\ 1 & 0 & -\frac{x}{\sqrt{4-x^2-y^2}} \\ 0 & 1 & -\frac{y}{\sqrt{4-x^2-y^2}} \end{matrix} \right\| =$$

$$= \left| \left(\frac{x}{\sqrt{4-x^2-y^2}}, \frac{y}{\sqrt{4-x^2-y^2}}, 1 \right) \right| =$$

$$= \sqrt{\frac{x^2+y^2+4-x^2-y^2}{4-x^2-y^2}} = \frac{2}{\sqrt{4-x^2-y^2}}$$

$$\Rightarrow \iint_S yz \, dS = \iint_D y \cdot \cancel{\sqrt{4-x^2-y^2}} \cdot \frac{2}{\cancel{\sqrt{4-x^2-y^2}}} \, dx\,dy =$$

$$= \left\{ \begin{matrix} \text{Polära} & 1 \le r \le 2 \\ \text{koord.} & \frac{\pi}{2} \le \theta \le \frac{5\pi}{4} \end{matrix} \right\} =$$

$$= 2 \int_{\pi/2}^{5\pi/4} \left(\int_1^2 r \sin\theta \, r \, dr \right) d\theta =$$

$$= 2 \cdot \left[-\cos\theta\right]_{\pi/2}^{\frac{5\pi}{4}} \cdot \left[\frac{r^3}{3}\right]_1^2 = 2 \cdot \frac{1}{\sqrt{2}} \cdot \frac{8-1}{3} = \frac{7\sqrt{2}}{3}$$

4. Parametrisering av $\mathcal{C}$:

$$x = t,\ 0 \leq t \leq 1 \Rightarrow y = 1 - t \Rightarrow$$

$$\Rightarrow z = \sqrt{1 - t^2 - (1-t)^2} = \sqrt{1 - t^2 - (1 - 2t + t^2)} = \sqrt{2t - 2t^2}$$

$$\therefore \mathcal{C} : \mathbf{r}(t) = (t, 1-t, \sqrt{2t - 2t^2}),\ 0 \leq t \leq 1$$

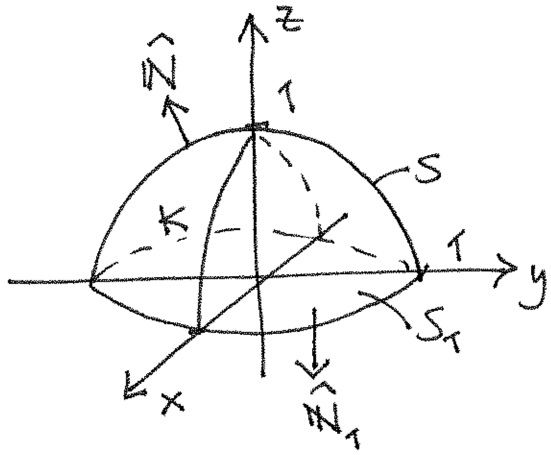
$$\int_{\mathcal{C}} z\, ds = \int_0^1 \sqrt{2t - 2t^2} \cdot \left| \left(1, -1, \frac{2 - 4t}{2\sqrt{2t - 2t^2}}\right) \right| dt =$$

$$= \int_0^1 \sqrt{2t - 2t^2} \cdot \sqrt{1 + 1 + \frac{(1 - 2t)^2}{2t - 2t^2}}\, dt =$$

$$= \int_0^1 \cancel{\sqrt{2t - 2t^2}} \cdot \frac{\sqrt{2 \cdot (2t - 2t^2) + (1 - 4t + 4t^2)}}{\cancel{\sqrt{2t - 2t^2}}}\, dt =$$

$$= \int_0^1 \sqrt{\cancel{4t} - \cancel{4t^2} + 1 - \cancel{4t} + \cancel{4t^2}}\, dt = 1$$

5. Komplettera med S_1 och använd Gauss sats



$$\iint_S \mathbb{F}\cdot\hat{\mathbb{N}}\,dS + \iint_{S_1}\mathbb{F}\cdot\hat{\mathbb{N}}_1\,dS =$$

$$= \oiint_{\partial K} \mathbb{F}\cdot\hat{\mathbb{N}}\,dS = \left\{\begin{matrix}\text{Gauss}\\ \text{sats}\end{matrix}\right\} =$$

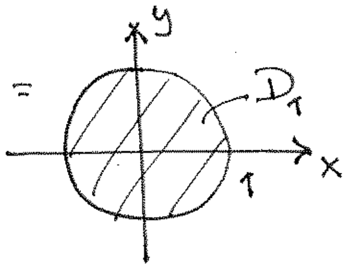
$$= \iiint_K \nabla\cdot\mathbb{F}\,dV = \iiint_K (z^2+0+3z^2)\,dV =$$

$$= 4\iiint_K z^2\,dV = \left\{ \begin{matrix} \text{Cylindriska} & 0\le r\le\sqrt{1-z} \\ \text{koord.} & 0\le\theta\le 2\pi \\ & 0\le z\le 1\end{matrix}\right\} =$$

$$= 4\int_0^1\left(\int_0^{2\pi}\left(\int_0^{\sqrt{1-z}} z^2\cdot r\,dr\right)d\theta\right)dz = 8\pi\int_0^1 z^2\left[\frac{r^2}{2}\right]_0^{\sqrt{1-z}}dz =$$

$$= 4\pi\int_0^1 z^2(1-z)\,dz = 4\pi\left[\frac{z^3}{3}-\frac{z^4}{4}\right]_0^1 = 4\pi\left(\frac{1}{3}-\frac{1}{4}\right)=\frac{\pi}{3}$$

$$\iint_{S_1}\mathbb{F}\cdot\hat{\mathbb{N}}_1\,dS = \left\{\begin{matrix} S_1: \mathbf{r}(x,y)=(x,y,0),\ (x,y)\in D_1 \\ \hat{\mathbb{N}}_1=(0,0,-1),\ dS=dA\end{matrix}\right\} =$$



$$= \iint_{D_1}(0,x^2,y^2)\cdot(0,0,-1)\,dA =$$

$$= -\iint_{D_1} y^2\,dA = \left\{\begin{matrix}\text{Polära}\\ \text{koord.}\end{matrix}\right\} = -\int_0^{2\pi}\left(\int_0^1 r^2\sin^2\theta\, r\,dr\right)d\theta =$$

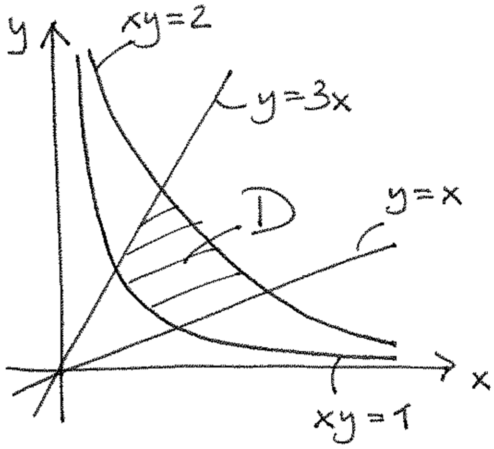
$$= \left\{\begin{matrix}\cos(2\theta)=\cos^2\theta-\sin^2\theta = \\ = 1-2\sin^2\theta\end{matrix}\right\} = -\int_0^{2\pi}\frac{1-\cos(2\theta)}{2}\,d\theta\cdot\int_0^1 r^3\,dr =$$

$$= -\frac{1}{2}\left[\theta - \frac{\sin(2\theta)}{2}\right]_0^{2\pi} \cdot \left[\frac{r^4}{4}\right]_0^1 = -\frac{1}{8}\cdot 2\pi = -\frac{\pi}{4}$$

$$\therefore \iint_S \mathbb{F}\cdot\hat{\mathbb{N}}\,dS = \iiint_K \nabla\cdot\mathbb{F}\,dV - \iint_{S_1}\mathbb{F}\cdot\hat{\mathbb{N}}_1\,dS =$$

$$= \frac{\pi}{3} - \left(-\frac{\pi}{4}\right) = \frac{4\pi + 3\pi}{12} = \frac{7\pi}{12}$$

6. Vi vill göra variabelbytet



$$u = \frac{y}{x}, \quad 1 \leq u \leq 3$$
$$v = xy, \quad 1 \leq v \leq 2$$
$\tilde{D}$

$$\Rightarrow \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} -\frac{y}{x^2} & \frac{1}{x} \\ y & x \end{vmatrix} = -\frac{y}{x} - \frac{y}{x} =$$

$$= -2\frac{y}{x} = -2u$$

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \frac{1}{\left|\frac{\partial(u,v)}{\partial(x,y)}\right|} = \frac{1}{|-2u|} = \{1 \leq u \leq 3\} = \frac{1}{2u}$$

Vill även veta vad xy^3 är i u och v:

$$u = \frac{y}{x} \Leftrightarrow y = xu$$

$$v = xy \Rightarrow v = x^2 u \Rightarrow \{x > 0 \text{ i } D\} \Rightarrow x = \sqrt{\frac{v}{u}}$$

$$\Rightarrow y = ux = u \cdot \sqrt{\frac{v}{u}} = \sqrt{uv}$$

$$\therefore \iint_D xy^3 \, dx\,dy = \iint_{\tilde{D}} \sqrt{\frac{v}{u}} \cdot (\sqrt{uv})^3 \cdot \frac{1}{2u} \, du\,dv =$$

$$= \frac{1}{2} \iint_{\tilde{D}} \frac{\sqrt{v}}{\cancel{\sqrt{u}}} \cdot \cancel{\sqrt{u}} \cdot \sqrt{v} \cdot \cancel{u} v \cdot \frac{1}{\cancel{u}} \, du\,dv =$$

$$= \frac{1}{2} \int_1^2 \left(\int_1^3 v^2 \, du \right) dv = \frac{1}{\cancel{2}} \cdot \cancel{2} \left[\frac{v^3}{3}\right]_1^2 = \frac{7}{3}$$

7. $$\frac{\partial u}{\partial x} = \frac{1}{z}\frac{\partial}{\partial x} f\left(\frac{yz}{x^2}, \frac{xz}{y^2}\right) =$$

$$= \frac{1}{z}\left(f_1 \cdot \left(-\frac{2yz}{x^3}\right) + f_2 \cdot \frac{z}{y^2}\right) = -\frac{2y}{x^3} f_1 + \frac{1}{y^2} f_2$$

$$\frac{\partial u}{\partial y} = \frac{1}{z}\frac{\partial}{\partial y} f\left(\frac{yz}{x^2}, \frac{xz}{y^2}\right) =$$

$$= \frac{1}{z}\left(f_1 \cdot \frac{z}{x^2} + f_2 \cdot \left(-\frac{2xz}{y^3}\right)\right) = \frac{1}{x^2} f_1 - \frac{2x}{y^3} f_2$$

$$\frac{\partial u}{\partial z} = \frac{\partial}{\partial z}\left(\frac{1}{z}\cdot f\left(\frac{yz}{x^2}, \frac{xz}{y^2}\right)\right) = -\frac{1}{z^2} f + \frac{1}{z}\frac{\partial}{\partial z} f\left(\frac{yz}{x^2}, \frac{xz}{y^2}\right) =$$

$$= -\frac{1}{z^2} f + \frac{1}{z}\left(f_1 \cdot \frac{y}{x^2} + f_2 \cdot \frac{x}{y^2}\right) =$$

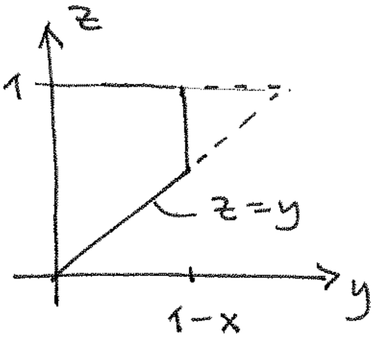
$$= -\frac{1}{z^2} f + \frac{y}{zx^2} f_1 + \frac{x}{zy^2} f_2$$

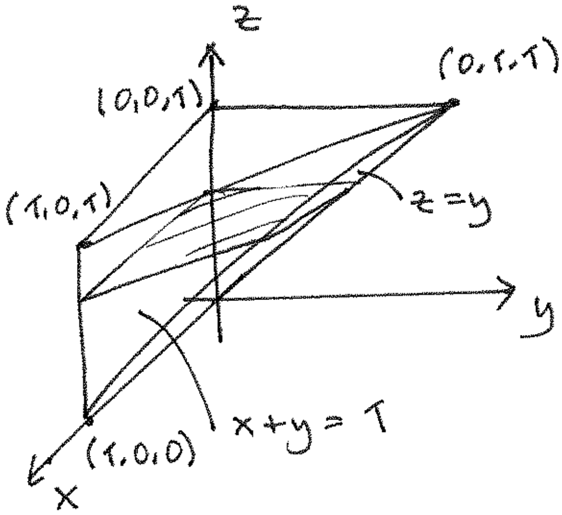
$$\therefore \; x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} + u =$$

$$= -\frac{2y}{x^2} f_1 + \frac{x}{y^2} f_2 + \frac{y}{x^2} f_1 - \frac{2x}{y^2} f_2 - \frac{1}{z} f + \frac{y}{x^2} f_1 + \frac{x}{y^2} f_2 + \frac{1}{z} f$$

$$= \frac{2y}{x^2} f_1 - \frac{2y}{x^2} f_1 + \frac{2x}{y^2} f_2 - \frac{2x}{y^2} f_2 + \frac{1}{z} f - \frac{1}{z} f = 0$$

8. Av integrationsgränserna framgår att kroppen är en pyramid med hörnen $(0,0,0)$, $(1,0,0)$, $(1,0,1)$, $(0,0,1)$ och $(0,1,1)$ (se figur)

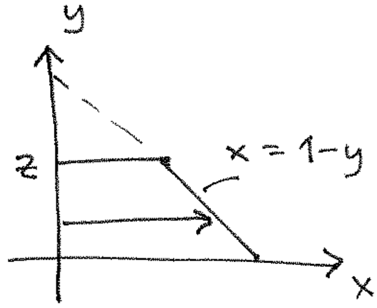




Vi kastar om integrationsordningen så att integreringen över z sker sist:

$$\int_0^1 \left(\int_0^{1-x} \left(\int_y^1 \frac{\sin(\pi z)}{z(2-z)}\, dz \right) dy \right) dx =$$

$$= \iiint_K \frac{\sin(\pi z)}{z(2-z)}\, dV =$$



$$= \int_0^1 \left(\int_0^z \left(\int_0^{1-y} \frac{\sin(\pi z)}{z(2-z)}\, dx \right) dy \right) dz =$$

$$= \int_0^1 \frac{\sin(\pi z)}{z(2-z)} \left(\int_0^z 1 - y\, dy \right) dz =$$

$$= \int_0^1 \frac{\sin(\pi z)}{z(2-z)} \cdot \left(z - \frac{z^2}{2} \right) dz = \int_0^1 \frac{\sin(\pi z)}{z(2-z)} \cdot \frac{z(2-z)}{2}\, dz =$$

$$= \frac{1}{2} \int_0^1 \sin(\pi z)\, dz = \frac{1}{2} \left[-\frac{\cos(\pi z)}{\pi} \right]_0^1 =$$

$$= \frac{1}{2} \cdot \left(-\frac{-1-1}{\pi} \right) = \frac{1}{\pi}$$