Korta lösningsförslag till tentamen MVE045, 14/1 2014

(1.) a) $\dfrac{1+\sin x + x}{x \arctan x} = \dfrac{1+\frac{\sin x}{x}+\frac{1}{x}}{\arctan x} \longrightarrow \dfrac{1}{\frac{\pi}{2}} = \dfrac{2}{\pi}$, $x \to \infty$

Svar: $\dfrac{2}{\pi}$

b) Sätt $f(x) = \dfrac{2}{3}\arctan\left(\dfrac{1}{3}\tan\dfrac{x}{2}\right)$

$$f'(x) = \frac{2}{3}\cdot\frac{1}{1+\left(\frac{1}{3}\tan\frac{x}{2}\right)^2}\cdot\frac{1}{3}\cdot\left(1+\tan^2\frac{x}{2}\right)\cdot\frac{1}{2} =$$

$$= \frac{1}{9+\tan^2\frac{x}{2}}\cdot\left(1+\tan^2\frac{x}{2}\right) = \left\{ \tan^2\frac{x}{2} = \frac{\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}} = \frac{1-\cos x}{1+\cos x} \right\} = \frac{1+\cos x+(1-\cos x)}{9(1+\cos x)+(1-\cos x)} = \frac{2}{10+8\cos x} =$$

$$= \frac{1}{5+4\cos x}$$

Svar: $\dfrac{1}{5+4\cos x}$

(2) $f(x) = \dfrac{x^3}{x^2-4} = \dfrac{x(x^2-4)+4x}{x^2-4} = x + \dfrac{4x}{x^2-4}$

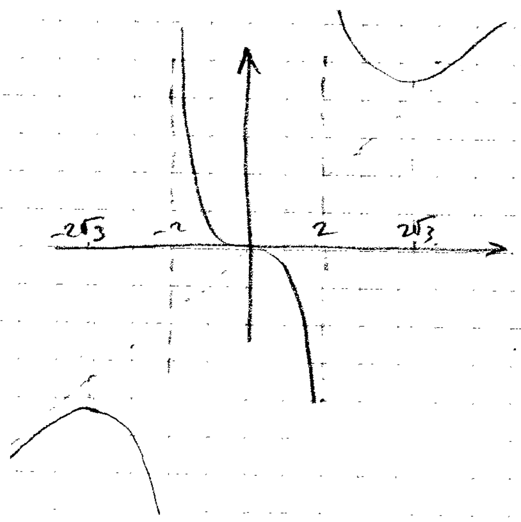
Vi ser att

- $D_f = D_{f'} = \{x \in \mathbb{R} : x \neq \pm 2\} = (-\infty,-2)\cup(-2,2)\cup(2,\infty)$
- $x = \pm 2$ är vertikala asymptoter och
  $y = x$ är en sned asymptot då $x \to \pm\infty$
- $f'(x) = 1 + \dfrac{4(x^2-4)-4x\cdot 2x}{(x^2-4)^2} = 1 + \dfrac{-4x^2-16}{(x^2-4)^2} = \ldots = \dfrac{x^2(x^2-12)}{(x^2-4)^2}$

$f'(x) = 0$ & $x \in D_f \iff x = \pm 2\sqrt{3}$ eller $x = 0$

teckenstudium

x		$-2\sqrt{3}$		-2		0		2		$2\sqrt{3}$	
$f'(x)$	+	0	−	⧙	−	0	−	⧙	−	0	+
$f(x)$	↗		↘	⧙	↘		↘	⧙	↘		↗



Lokalt min för $x = 2\sqrt{3}$

Lokalt max för $x = -2\sqrt{3}$

inga globala min eller max

③ a) $\sin x = x - \frac{x^3}{6} + \frac{x^5}{120} + O(x^7)$

$\sin(\sin x) = \left(x - \frac{x^3}{6} + \frac{x^5}{120} + O(x^7)\right) - \frac{1}{6}\left(x - \frac{x^3}{6}\right)^3 +$

$+ \frac{1}{120}x^5 + O(x^7) = x + \left(-\frac{1}{6} - \frac{1}{6}\right)x^3 + \left(\frac{1}{120} + \frac{1}{2}\cdot\frac{1}{6} + \right.$

$\left.\frac{1}{120}\right)x^5 + O(x^7) = x - \frac{1}{3}x^3 + \frac{1}{10}x^5 + O(x^7)$

$\arctan x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 + O(x^7)$

Alltså: $\sin(\sin x) - \arctan x = \left(\frac{1}{10} - \frac{1}{5}\right)x^5 + O(x^7) =$

$= -\frac{1}{10}x^5 + O(x^7)$ och $\frac{\sin(\sin x) - \arctan x}{x^5} \to -\frac{1}{10},\ x \to 0$

Svar: $-\frac{1}{10}$

b) $\sin t = t - \frac{t^3}{6} + \frac{t^5}{120} + O(t^7)$, $t \to 0$

Insättning av $t = \frac{x^2}{2}$ ger $\sin\frac{x^2}{2} = \frac{x^2}{2} - \frac{x^6}{48} + \frac{x^{10}}{16\cdot 120} + O(x^4)$

Svar $\frac{x^2}{2} - \frac{x^4}{48} + \frac{x^{10}}{16\cdot 120} + x^4 B(x)$

④ a) $z^2 - (3+2i)z + 5 + i = 0$ ger

$\left(z - \left(\frac{3}{2} + i\right)\right)^2 = +\left(\frac{3}{2} + i\right)^2 - 5 - i = -\frac{15}{4} + 2i$

Sätt $a + ib = z - \left(\frac{3}{2} + i\right)$, $a, b \in \mathbb{R}$

Då gäller

$$\begin{cases} a^2 - b^2 = -\frac{15}{4}, \quad 2ab = 2 \\ a^2 + b^2 = \sqrt{\left(-\frac{15}{4}\right)^2 + 2^2} = \frac{1}{4}\sqrt{289} = \frac{17}{4} \end{cases}$$

Detta ger $a^2 = \frac{1}{4}$ dvs $a = \pm\frac{1}{2}$ och $b = \pm 2$

Vi får $z_{1,2} = \frac{3}{2} + i \pm \left(\frac{1}{2} + 2i\right) = \begin{cases} 2 + 3i \\ 1 - i \end{cases}$

Svar: $2 + 3i,\ 1 - i$

b) $4y'' + 4y' + y = 0$ har kar. pol $4r^2 + 4r + 1 = 4\left(r^2 + r + \frac{1}{4}\right) =$

$= 4\left(r + \frac{1}{2}\right)^2$ och allmän ger homogenlösningen

är $y(x) = (A + Bx)e^{-\frac{x}{2}}$.

Villkoren $y(0) = 2$, $y'(0) = 0$ ger

$$\begin{cases} 2 = A \\ 0 = \left(-\frac{1}{2}A + B\right) \end{cases} \text{ dvs } \begin{cases} A = 2 \\ B = 1 \end{cases}$$

Svar $y(x) = (2 + x)e^{-\frac{x}{2}}$

5 a) $\int x^3 \arctan x \, dx = \{PI\} =$

$$= \frac{x^4}{4} \cdot \arctan x - \int \frac{x^4}{4} \cdot \frac{1}{1+x^2} \, dx =$$

$$= \frac{x^4}{4} \cdot \arctan x - \frac{1}{4} \int \frac{x^4+x^2-x^2-1+1}{x^2+1} \, dx =$$

$$= \frac{x^4}{4} \cdot \arctan x - \frac{1}{4} \int \left(x^2 - 1 + \frac{1}{1+x^2}\right) dx =$$

$$= \left(\frac{x^4}{4} - \frac{1}{4}\right) \arctan x - \frac{x^3}{12} + \frac{x}{4} + C$$

Svar: $\left(\frac{x^4}{4} - \frac{1}{4}\right) \arctan x - \frac{x^3}{12} + \frac{x}{4} + C$

b) $\int_0^1 (1-x^3)^{50} \cdot x^5 \, dx = \{ 1-x^3 = t, \ -3x^2 dx = dt,$

$x=0 \leftrightarrow t=1, \ x=1 \leftrightarrow t=0 \} =$

$$= -\frac{1}{3} \int_1^0 t^{50} \cdot (1-t) \, dt = \frac{1}{3} \int_0^1 (t^{50} - t^{51}) \, dt =$$

$$= \frac{1}{3} \left[\frac{t^{51}}{51} - \frac{t^{52}}{52} \right]_0^1 = \frac{1}{3} \cdot \left(\frac{1}{51} - \frac{1}{52} \right) = \frac{1}{3 \cdot 51 \cdot 52}$$

Svar: $\frac{1}{3 \cdot 51 \cdot 52}$