OPTIONS AND MATHEMATICS (CTH[mve095])
MATEMATIK OCH OPTIONER (GU[man690])

ASSIGNMENTS 2007
Must be handed in at the latest May 16 at 14^{45}.
Please hand in solutions of at most 6 out of the following 12 problems; each problem gives maximum 0.2 credit points.

The problems numbered 2, 3, 5, 7a, and 8a are standard problems, where the computer is of no help. The other problems are based on MATLAB. Each figure must be simple to understand and with defined label and axes. No parts of the MATLAB figures are allowed to be written by hand. Good luck!

Problem 1 (MATLAB; title('Problem 1'), legend('Bull spread')) A European derivative pays the amount $g(S(T))$ at time of maturity T, where $g(s) = (s-2)^+ - (s-4)^+$, $0 \le s \le 6$. Plot the graph of the payoff function $y = g(s)$.

Problem 2 {Handwritten solution} (Dominance Principle) Suppose $t < T$ and $c(t, S(t), K, T) = 3.14$. The price of a European derivative at time t is N units of currency and it pays at the maturity date T the amount

$$N + \alpha N(S(T) - K)^+.$$

If $N \neq 0$, show that

$$\alpha = (1 - e^{-r(T-t)})/3.14.$$

Problem 3 {Handwritten solution} (Dominance Principle) Below θ_i, $i = 0, 1, ..., n$, are positive real numbers such that $\Sigma_0^n \theta_i = 1$.

(a) Suppose $f : I \to \mathbf{R}$ is convex and $x_0, ..., x_n \in I$, where I is a subinterval of $\mathbf{R}$. Show that

$$f(\Sigma_0^n \theta_i x_i) \le \Sigma_0^n \theta_i f(x_i).$$

(Hint: Use induction on n.)

(b) Suppose $a_i > 0$, $i = 0, 1, ..., n$. Show that

$$\Pi_0^n a_i^{\theta_i} \le \Sigma_0^n \theta_i a_i.$$

(Hint: If $f :]0, \infty[\to \mathbf{R}$ is differentiable and f' increasing, then f is convex.)

(c) Suppose $K > 0$ and $t \le t_0 \le ... \le t_n \le T$. Two European derivatives with maturity date T have the payoffs

$$Y_1 = (\Pi_0^n S(t_i)^{\theta_i} - K)^+$$

and

$$Y_2 = (\Sigma_0^n \theta_i S(t_i) - K)^+,$$

respectively. Show that

$$\Pi_{Y_1}(t) \le \Pi_{Y_2}(t) \le c(t, S(t), K, T).$$

Problem 4 (MATLAB; title('Problem 4'), legend('Monte Carlo computation of Asian option price')) Consider the binomial model with $T = 44$, $u = -d = 0.038$, $r = 0.002$, and $S(0) = 100$. Suppose an Asian option pays the amount

$$Y = (101 - \frac{1}{43} \sum_{k=1}^{43} S(k))^+$$

to its holder at time 44. Use the Monte Carlo method with n simulations to find an approximation $\Pi_Y^n(0)$ of the option price $\Pi_Y(0)$ at time 0. Plot the sequence $\Pi_Y^n(0)$, $n = 20000$, 22500, $25000, ..., 100000$.

Problem 5 {Handwritten solution} Suppose X is is a random variable such that

$$P[X = -1] = P[X = 1] = \frac{1}{2}.$$

Find all $\lambda \in \mathbf{R}$ such that

$$E\left[(a + \lambda bX)^4\right] \le (E\left[(a + bX)^2\right])^2$$

for all a, $b \in \mathbf{R}$. (Hint: $X = X^3$ and $X^2 = X^4 = 1$.)

Problem 6 (MATLAB; title('Problem 6'), legend('$N(0,\sigma)$ density for various σ')) Plot the graph of the function $z = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}}$, $-10 \leq x \leq 10$, $1 \leq \sigma \leq 5$.

Problem 7 (MATLAB; title('Problem 7'), legend('Sample data','N(0,1) density')) Suppose U has a uniform distribution in the unit interval $[0,1]$. (a) {Handwritten solution} Compute $\mu = E\left[e^{\sqrt{U}}\right]$ and σ =Var($e^{\sqrt{U}}$) (b) Let U_{ij}, $i = 1, ..., n$, $j = 1, ..., M$, be independent observations on U and set

$$S_j = \frac{1}{\sigma\sqrt{n}} \sum_{i=1}^{n} (e^{\sqrt{U_{ij}}} - \mu),\ \ j = 1, ..., M.$$

Plot a bar graph or histogram of $(S_j)_{j=1}^{M}$ taking $n = 500$ and $M = 10000$. Scale so that the area of the histograph adds up to one. (c) Plot the $N(0,1)$ density function in the same figure. (d) Finally, write the numerical values of μ and σ in the figure.

Problem 8 (MATLAB; title('Problem 8'), legend('Lognormal density of a stock price')) Consider the geometric Brownian motion modell of a stock price process $(S(t))_{t\geq 0}$, where

$$S(t) = S(0)e^{(\mu - \frac{\sigma^2}{2})t + \sigma W(t)}.$$

(a) {Handwritten solution} Suppose $t > 0$. Find the density function f of the random variable $S(t)$. (b) Suppose $t = 1$, $S(0) = 1$, and $\mu = 0.05$. Plot the graph of the density function f for the values $\sigma = 0.3$ and $\sigma = 0.5$.

Problem 9 (MATLAB; title('Problem 9'), legend('50 asset prices')) Consider the geometric Brownian motion model of a stock price process $(S(t))_{0\leq t\leq T}$, where

$$S(t) = S(0)e^{(\mu - \frac{\sigma^2}{2})t + \sigma W(t)}$$

Plot 50 realizations of the stock price process for $S(0) = 1$, $\mu = 0.05$, $\sigma = 0.5$, and $T = 1$.

Problem 10 (MATLAB) Define a MATLAB program with output arguments C, $Cdelta$, P, and $Pdelta$ representing, respectively, the European call price, call delta, European put price, and put delta values. Let the input arguments be $\tau = T - t$, $S(0)$, K, r, and σ. Check the values of C against the call prices on page 83 in the book "Introduction to the Black-Scholes Theory". (a) Complete with values of P (b) Complete with values of $Cdelta$.

Problem 11 (MATLAB; title('Problem 11'), legend('Theoretic value','Actual value') Consider the geometric Brownian motion model of a stock price process $(S(t))_{0 \leq t \leq T}$, where

$$S(t) = S(0)e^{(\mu - \frac{\sigma^2}{2})t + \sigma W(t)}$$

Suppose $h = T/N$ and $i = nh$, $n = 0, ..., N$. Define $cash(t_0)$ by the equation

$$c(t_0, S(0), K, T) = \Delta_C(t_0)S(t_0) + cash(t_0)$$

where $\Delta_C(t) = c'_s(t, S(t), K, T)$. Set

$$\Pi_1 = \Delta_C(t_0)S(t_1) + cash(t_0)e^{rh}$$

where r is the interest rate. Next define $cash(t_1)$ by the equation

$$\Pi_1 = \Delta_C(t_1)S(t_1) + cash(t_1)$$

and

$$\Pi_2 = \Delta_C(t_1)S(t_2) + cash(t_1)e^{rh}$$

and continue the process until maturity T. Plot the call prices $c(t_i, S(t_i), K, T)$, $i = 0, ..., N$, and the portfolio values Π_i, $i = 0, ..., N$ in the same figure when $t = 0$, $S(0) = 100$, $K = 105$, $T = 0.5$, $\mu = 0.05$, $r = 0.03$, $\sigma = 0.35$, and $N = 130$. Finally, find $\Pi_N - (S(T) - K)^+$.

Problem 12 (MATLAB; title('Problem 12'), legend('European put price', 'max($strike - s, 0$)','American put price')) (Black-Scholes model) Define a MATLAB program for the price of an American put using the algorithm on page 85 in "An Introduction to the Black-Scholes Theory". (a) Plot the graphs of the functions $y = p(t, s, K, T)$, $y = \max(K - s, 0)$, and $y = P(t, s, K, T)$, in the stock price interval $39 \leq s \leq 46$, when $t = 0$, $K = 45$, $T = 1/12$, $r = \ln(1.05)$, and $\sigma = 0.2$. The graphs must be given in the same figure.