SOLUTIONS: OPTIONS AND MATHEMATICS
(CTH[mve095], GU[MAN690])

January 19, 2008, morning (4 hours), v
No aids.
Examiner: Christer Borell, telephone number 0705292322
Each problem is worth 3 points.

1. (Black-Scholes model) A derivative of European type pays the amount

$$Y = S(T) + \frac{1}{S(T)}$$

at time of maturity T. Find $\Pi_Y(t)$ for all $0 \leq t < T$.

Solution. We have

$$\Pi_Y(t) = \Pi_{S(T)}(t) + \Pi_{\frac{1}{S(T)}}(t).$$

Here, if $\tau = T - t$, $s = S(t)$, and $G \in N(0,1)$,

$$\Pi_{S(T)}(t) = e^{-r\tau} E\left[se^{(r-\frac{\sigma^2}{2})\tau + \sigma\sqrt{\tau}G} \right]$$

$$= se^{-\frac{\sigma^2}{2}\tau} E\left[e^{\sigma\sqrt{\tau}G} \right] = se^{-\frac{\sigma^2}{2}\tau} e^{\frac{\sigma^2}{2}\tau} = s.$$

Moreover,

$$\Pi_{\frac{1}{S(T)}}(t) = e^{-r\tau} E\left[\frac{1}{se^{(r-\frac{\sigma^2}{2})\tau + \sigma\sqrt{\tau}G}} \right]$$

$$= e^{-r\tau} \frac{e^{-(r-\frac{\sigma^2}{2})\tau}}{s} E\left[e^{-\sigma\sqrt{\tau}G} \right]$$

$$= \frac{e^{-(2r-\frac{\sigma^2}{2})\tau}}{s} e^{\frac{1}{2}\sigma^2\tau} = \frac{1}{s} e^{(\sigma^2 - 2r)\tau}$$

and it follows that

$$\Pi_Y(t) = S(t) + \frac{1}{S(t)} e^{(\sigma^2 - 2r)\tau}.$$

2. (Binomial model) Suppose $d = -u$ and $e^r = \frac{1}{2}(e^u + e^d)$. A financial derivative of European type has the maturity date $T = 4$ and payoff $Y = f(X_1 + X_2 + X_3 + X_4)$, where $f(x) = 1$ if $x \in \{4u, 0, -4u\}$ and $f(x) = -1$ if $x \in \{2u, -2u\}$. Show that $\Pi_Y(0) = 0$.

Solution. It follows that $d < r < u$ and

$$q_u = \frac{e^r - e^d}{e^u - e^d} = \frac{e^u - e^r}{e^u - e^d} = q_d.$$

Hence $q_u = q_d = \frac{1}{2}$. Furthemore,

$$\Pi_Y(0) = e^{-4r} \sum_{k=0}^{4} \left(\begin{array}{c} 4 \\ k \end{array} \right) q_u^k q_d^{4-k} f(ku + (4-k)d)$$

$$= e^{-4r} \sum_{k=0}^{4} \left(\begin{array}{c} 4 \\ k \end{array} \right) q_u^k q_d^{4-k} f((2k-4)u)$$

$$= e^{-4r} (\frac{1}{2})^4 (1 - 4 + 6 - 4 + 1) = 0.$$

3. (Black-Scholes model) Suppose $T > 0$, $N \in \mathbf{N}_+$, $h = \frac{T}{N}$, and $t_n = nh$, $n = 0, ..., N$, and consider a derivative of European type paying the amount $Y = \sum_{n=0}^{N-1} (\ln \frac{S(t_{n+1})}{S(t_n)})^2$ at time of maturity T. Find $\Pi_Y(0)$.

Solution. First consider a derivative paying the amount $Y_n = (\ln \frac{S(t_{n+1})}{S(t_n)})^2$ at time T. Since Y_n is known at time t_{n+1}, $\Pi_{Y_n}(t_{n+1}) = Y_n e^{-r(T-t_{n+1})}$. Note that

$$S(t_{n+1}) = S(t_n) e^{(\mu - \frac{\sigma^2}{2})h + \sigma(W(t_{n+1}) - W(t_n))}$$

where $W(t_{n+1}) - W(t_n) \in N(0, h)$. Thus, if $G \in N(0, 1)$,

$$\Pi_{Y_n}(t_n) = e^{-rh} E \left[e^{-r(T-t_{n+1})} \left\{ (r - \frac{\sigma^2}{2})h + \sigma\sqrt{h}G \right\}^2 \right]$$

$$= e^{-r(T-t_n)} \left\{ (r - \frac{\sigma^2}{2})^2 h^2 + \sigma^2 h \right\}$$

and since the expression for $\Pi_{Y_n}(t_n)$ is known at time 0,

$$\Pi_{Y_n}(0) = e^{-t_n h} e^{-r(T-t_n)} \left\{ (r - \frac{\sigma^2}{2})^2 h^2 + \sigma^2 h \right\}$$

$$= e^{-rT} \left\{ (r - \frac{\sigma^2}{2})^2 h^2 + \sigma^2 h \right\}.$$

Now it follows that

$$\Pi_Y(0) = \sum_{n=0}^{N-1} \Pi_{Y_n}(0) = N e^{-rT} \left\{ (r - \frac{\sigma^2}{2})^2 h^2 + \sigma^2 h \right\}$$

$$= T e^{-r\tau} \left\{ \sigma^2 + h(r - \frac{\sigma^2}{2})^2 \right\}.$$

4. Derive the delta of a European call in the Black-Scholes model. Recall that the call price equals $s\Phi(d_1) - Ke^{-r\tau}\Phi(d_2)$, where $s = S(t)$, $\tau = T - t > 0$, and

$$d_1 = \frac{\ln \frac{s}{K} + (r + \frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}} = d_2 + \sigma\sqrt{\tau}.$$

5. Consider the binomial model in one period and assume $d < r < u$. A derivative pays the amount $Y = f(X)$ at time 1. Find a portfolio which replicates the derivative at time 0.