OPTIONS AND MATHEMATICS (CTH[$mve095$] , GU[$MMA700$])
ASSIGNMENTS 2008
Must be handed in at the latest May 15 at 11^{45}.
Please hand in solutions of at most 8 out of the following 10 problems; each problem gives maximum 0.125 credit points at the final examination in May 2008.

The problems numbered 2, 3, 4, 5, 7, 8(a) and 10 are of standard type and you may hand in handwritten solutions (you need a computer to solve Exercises 7(b), 7(c), and 10). The MATLAB figures in Problems 1, 6, 8(b), and 9 must be simple to understand and with defined label and axes. No parts of the MATLAB figures are allowed to be written by hand. Good luck!

Problem 1 (MATLAB; title('Problem 1'), legend('Bull spread')) A European derivative pays the amount $g(S(T))$ at time of maturity T, where $g(s) = (s-2)^+ - (s-4)^+$, $0 \leq s \leq 6$. Plot the graph of the payoff function $y = g(s)$.

Problem 2 Let $t < T$ and $N \in \mathbf{N}_+$. Set $\tau = T - t$, $h = \tau/N$, and $t_n = t + nh$, $n = 0, ..., N$. Furthermore, suppose $S = (S(t))_{t \geq 0}$ is a stock price process. A financial contract has the following description: at each point of time t_{n-1} the holder of the contract gets a forward contract on S with delivery date t_n and at time t_n the holder's saving account adds the amount $S(t_n) - S_{for}^{t_n}(t_{n-1})$ for $n = 1, ..., N$. Prove that the sum of the depositions will grow to the amount $S(T) - S_{for}^{T}(t)$ at time T.

Problem 3 (Dominance Principle) Below θ_i, $i = 0, 1, ..., n$, are positive real numbers such that $\Sigma_0^n \theta_i = 1$.

(a) Suppose $f : I \to \mathbf{R}$ is convex and $x_0, ..., x_n \in I$, where I is a subinterval of $\mathbf{R}$. Show that

$$f(\Sigma_0^n \theta_i x_i) \leq \Sigma_0^n \theta_i f(x_i).$$

(Hint: Use induction on n.)

(b) Suppose $a_i > 0$, $i = 0, 1, ..., n$. Show that

$$\Pi_0^n a_i^{\theta_i} \leq \Sigma_0^n \theta_i a_i.$$

(Hint: If $f :]0, \infty[\to \mathbf{R}$ is differentiable and f' increasing, then f is convex.)

(c) Suppose $K > 0$ and $t \leq t_0 \leq ... \leq t_n \leq T$. Two European derivatives with maturity date T have the payoffs

$$Y_1 = (\Pi_0^n S(t_i)^{\theta_i} - K)^+$$

and

$$Y_2 = (\Sigma_0^n \theta_i S(t_i) - K)^+,$$

respectively. Show that

$$\Pi_{Y_1}(t) \leq \Pi_{Y_2}(t) \leq c(t, S(t), K, T).$$

Problem 4 Suppose the random variable X is positive with probability one and $\ln X \in N(0, 1)$. (a) Find the density function f of X. (b) Set

$$g(x) = \begin{cases} f(x)(1 + \sin(2\pi \ln x)), & x > 0, \\ 0, \text{ if } & x \leq 0. \end{cases}$$

Show that $g(x) \geq 0$ and

$$\int_0^\infty x^k g(x) dx = \int_0^\infty x^k f(x) dx$$

for all $k \in \mathbf{N}$. (Therefore there exists a random variable Y, with a different distribution function than X, such that $E[p(Y)] = E[p(X)]$ for every polynomial $p(x)$.)

Problem 5 A random variable X has the density function $f(x) = \frac{x^2}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$, $x \in \mathbf{R}$. Find the characteristic function $c_X(\xi) = E\left[e^{i\xi X}\right]$, $\xi \in \mathbf{R}$.

Problem 6 (MATLAB; title('Problem 6'), legend('$N(0, \sigma)$ density for various σ')) Plot the graph of the function $z = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}}$, $-10 \leq x \leq 10$, $1 \leq \sigma \leq 5$.

Problem 7 Suppose $f(x) = \sin(x+1)$, $x \in \mathbf{R}$, and let $G_1, ..., G_n \in N(0,1)$ be independent.

(a) Find the value of the integral

$$I = \int_{-\infty}^{\infty} f(x) e^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}}.$$

(b) Find approximations of I using the estimates

$$MC_1 = \frac{1}{n} \sum_{k=1}^{n} f(G_k)$$

for $n = 100, 1000, 10000$, and 100000. (Hint: Use MATLAB)

(c) Find approximations of I using the estimates

$$MC_2 = \frac{1}{2n} \sum_{k=1}^{n} (f(G_k) + f(-G_k))$$

for $n = 100, 1000, 10000$, and 100000. (Hint: Use MATLAB)

Problem 8 (MATLAB; title('Problem 8'), legend('Lognormal density of a stock price')) Consider the geometric Brownian motion modell of a stock price process $(S(t))_{t \geq 0}$, where

$$S(t) = S(0) e^{(\mu - \frac{\sigma^2}{2})t + \sigma W(t)}.$$

(a) {Handwritten solution} Suppose $t > 0$. Find the density function f of the random variable $S(t)$. (b) Suppose $t = 1$, $S(0) = 1$, and $\mu = 0.05$. Plot the graph of the density function f for the values $\sigma = 0.3$ and $\sigma = 0.5$.

Problem 9 (MATLAB; title('Problem 9'), legend('50 log-Brownian asset prices')) Consider the geometric Brownian motion model of a stock price process $(S(t))_{0 \leq t \leq T}$, where

$$S(t) = S(0) e^{(\mu - \frac{\sigma^2}{2})t + \sigma W(t)}$$

Plot 50 realizations of the stock price process for $S(0) = 1$, $\mu = 0.05$, $\sigma = 0.5$, and $T = 1$.

Problem 10 Let c, $cdelta$, and p be, respectively, the European call price, European call delta, and European put price. Suppose the input arguments are $\tau = T - t$, $S(0)$, K, r, and σ. Check the values of c against the call prices on page 88 in the book "Introduction to the Black-Scholes Theory", (printed in February 08). (a) Complete with values of p (b) Complete with values of $cdelta$. (Hint: Use MATLAB)