SOLUTIONS
OPTIONS AND MATHEMATICS
(CTH[mve095], GU[MMA700])

August 30, 2008, morning (4 hours), V
No aids.
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Each problem is worth 3 points.

1. (Black-Scholes model) A derivative of European type pays the amount $Y = \frac{S(T)}{S(T/2)}$ at time of maturity T. Find $\Pi_Y(0)$.

Solution. For any $t \in [0, T]$ and real number a, $\Pi_{aS(T)}(t) = aS(t)$ and, hence,

$$\Pi_Y(T/2) = \Pi_{\frac{1}{S(T/2)}S(T)}(T/2) = \frac{1}{S(T/2)}\Pi_{S(T)}(T/2)$$

$$= \frac{1}{S(T/2)}S(T/2) = 1.$$

Accordingly from this,

$$\Pi_Y(0) = e^{-\frac{rT}{2}}.$$

2. Suppose $Z = (Z_1(t), Z_2(t))_{t \geq 0}$ is a standard Brownian motion in the plane. Find $E\left[\sqrt{Z_1^2(t) + Z_2^2(t)}\right]$ if $t \geq 0$.

Solution. Let $t \geq 0$ be fixed. Since $(Z_1(t), Z_2(t))$ has the same distribution as $\sqrt{t}(Z_1(1), Z_2(1))$,

$$E\left[\sqrt{Z_1^2(t) + Z_2^2(t)}\right] = E\left[\sqrt{t(Z_1^2(1) + Z_2^2(1))}\right]$$

$$= \sqrt{t}\iint\limits_{\mathbf{R}^2} \sqrt{x^2+y^2}e^{-\frac{x^2+y^2}{2}}\frac{dxdy}{2\pi} = \left[\begin{array}{c}\text{polar} \\ \text{coordinates}\end{array}\right]$$

$$= \sqrt{t}\int_0^\infty \int_0^{2\pi} r^2 e^{-\frac{r^2}{2}} \frac{dr d\theta}{2\pi} = \sqrt{t}\int_0^\infty r^2 e^{-\frac{r^2}{2}} dr = \left[\begin{array}{c} \text{partial} \\ \text{integration} \end{array} \right]$$

$$= \sqrt{t}\int_0^\infty e^{-\frac{r^2}{2}} dr = \sqrt{\frac{\pi t}{2}}.$$

3. (Black-Scholes model) Suppose K is a positive real number and consider a simple derivative of European type with the payoff

$$Y = (\frac{1}{S(T)} - K)^+$$

at time of maturity T. Moreover, suppose $0 < t^* < T$ and $0 < \delta < 1$. Find $\Pi_Y(0)$ if the stock pays the dividend $\delta S(t^*-)$ at time t^*.

Solution. Let $s = S(0)$ and suppose $G \in N(0,1)$. We have

$$\Pi_Y(0) = e^{-rT} E\left[(\frac{1}{(1-\delta)s e^{(r-\frac{\sigma^2}{2})T + \sigma\sqrt{T}G}} - K)^+ \right]$$

$$= \frac{e^{-rT}}{(1-\delta)s} E\left[(e^{-(r-\frac{\sigma^2}{2})T - \sigma\sqrt{T}G} - L)^+ \right]$$

where $L = (1-\delta)sK$. Here

$$E\left[(e^{-(r-\frac{\sigma^2}{2})T - \sigma\sqrt{T}G} - L)^+ \right] = \int_{-\infty}^{-\frac{1}{\sigma\sqrt{T}}(\ln L + (r-\frac{\sigma^2}{2})T)} (e^{-(r-\frac{\sigma^2}{2})T - \sigma\sqrt{T}x} - L) e^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}}$$

$$= e^{(\sigma^2 - r)T} \int_{-\infty}^{-\frac{1}{\sigma\sqrt{T}}(\ln L + (r-\frac{\sigma^2}{2})T)} e^{-\frac{1}{2}(x+\sigma\sqrt{T})^2 \frac{dx}{\sqrt{2\pi}} - L\Phi(-\frac{1}{\sigma\sqrt{T}}(\ln L + (r-\frac{\sigma^2}{2})T))}$$

$$= e^{(\sigma^2 - r)T}\Phi(-\frac{1}{\sigma\sqrt{T}}(\ln L + (r - \frac{3}{2}\sigma^2)T)) - L\Phi(-\frac{1}{\sigma\sqrt{T}}(\ln L + (r - \frac{\sigma^2}{2})T)).$$

Thus

$$\Pi_Y(0) = \frac{e^{(\sigma^2 - 2r)T}}{(1-\delta)s}\Phi(-\frac{1}{\sigma\sqrt{T}}(\ln L + (r - \frac{3}{2}\sigma^2)T)) - e^{-rT}K\Phi(-\frac{1}{\sigma\sqrt{T}}(\ln L + (r - \frac{\sigma^2}{2})T)).$$

4. Prove that there exists an arbitrage portfolio in the single-period binomial model if and only if

$$r \notin \,]d, u[\,.$$

5. (Black-Scholes model) Consider a European call on a stock with price process $(S(t))_{t\geq 0}$. If K denotes strike price and T time of maturity, the Black-Scholes price of the call at time $t < T$ equals

$$c(t, S(t), K, T)) = s\Phi(d_1) - Ke^{-r\tau}\Phi(d_2),$$

where $\tau = T - t$ and

$$d_1 = d_2 + \sigma\sqrt{\tau} = \frac{1}{\sigma\sqrt{\tau}}(\ln\frac{S(t)}{K} + (r + \frac{\sigma^2}{2})\tau).$$

(a) Find the delta of the call.

(b) How is the call price formula changed if the stock price pays the dividend D at time $t^* \in \,]t, T[\,$, where D is a fixed amount known at time t?