SOLUTIONS
OPTIONS AND MATHEMATICS
(CTH[mve095], GU[MMA700])

January 16, 2010, morning (4 hours), v
No aids.
Examiner: Christer Borell, telephone number 0705292322
Each problem is worth 3 points.

1. (Binomial model, T periods) Set

$$Y = \frac{1}{T}\sum_{t=1}^{T} \ln \frac{S(t)}{S(t-1)}.$$

Prove that $E[Y] = d + p_u(u-d)$ and $\mathrm{Var}(Y) = \frac{1}{T}p_u(1-p_u)(u-d)^2$.

Solution. Using standard notation

$$S(t) = S(t-1)e^{X_t},\ t = 1, ..., T$$

where $X_1, ..., X_T$ are independent and

$$\begin{cases} P[X_t = u] = p_u \\ P[X_t = d] = p_d. \end{cases}$$

Note that

$$E[X_t] = p_u u + p_d d = d + p_u(u-d),$$
$$E\left[X_t^2\right] = p_u u^2 + p_d d^2,$$

and

$$\mathrm{Var}(X_t) = p_u u^2 + p_d d^2 - (p_u u + p_d d)^2$$
$$= p_u(1-p_u)(u^2+d^2) - 2p_u p_d ud = p_u(1-p_u)(u-d)^2.$$

Now since

$$Y = \frac{1}{T}\sum_{t=1}^{T} X_t$$

we have that

$$E[Y] = \frac{1}{T}\sum_{t=1}^{T} E[X_t] = d + p_u(u-d)$$

and

$$\mathrm{Var}(Y) = \frac{1}{T^2}\sum_{t=1}^{T} \mathrm{Var}(X_t) = \frac{1}{T}p_u(1-p_u)(u-d)^2.$$

2. (Black-Scholes model) Let $a, K, T > 0$ be given numbers and consider a simple derivative of European type with time of maturity T and payoff K if $S(T) < a$ and payoff 0 if $S(T) \geq a$. (a) Find the price of the derivative at time $t < T$. (b) Find the delta of the derivative at time $t < T$. (c) Find the vega of the derivative at time $t < T$.

Solution. (a) Set $\tau = T - t$ and let $G \in N(0,1)$. The price of the derivative at time t equals $\pi(t) = v(t, S(t))$, where

$$v(t,s) = e^{-r\tau}E\left[K1_{]0,a[}(se^{(r-\frac{\sigma^2}{2})\tau+\sigma\sqrt{\tau}G})\right]$$

$$= e^{-r\tau}KP\left[se^{(r-\frac{\sigma^2}{2})\tau+\sigma\sqrt{\tau}G} < a\right] = e^{-r\tau}KP\left[G < \frac{\ln\frac{a}{s} - (r-\frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}}\right]$$

$$= e^{-r\tau}K\Phi(\frac{\ln\frac{a}{s} - (r-\frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}}).$$

Hence

$$\pi(t) = e^{-r\tau}K\Phi(\frac{\ln\frac{a}{S(t)} - (r-\frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}}).$$

(b) Let $\varphi = \Phi'$. The delta at time t is given by $\frac{\partial v}{\partial s}_{|s=S(t)}$, where

$$\frac{\partial v}{\partial s} = e^{-r\tau}K\varphi(\frac{\ln\frac{a}{s} - (r-\frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}})\frac{-1}{\sigma\sqrt{\tau}s}.$$

Thus the delta equals

$$-\frac{e^{-r\tau}K}{\sigma\sqrt{\tau}S(t)}\varphi(\frac{\ln\frac{a}{S(t)} - (r-\frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}}).$$

(c) The vega at time t is given by $\frac{\partial v}{\partial \sigma}(t, S(t))$ and equals

$$\begin{aligned} & e^{-r\tau} K\varphi\left(\frac{\ln \frac{a}{S(t)} - (r - \frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}}\right)\left\{-\frac{\ln \frac{a}{S(t)} - r\tau}{\sigma^2\sqrt{\tau}} + \frac{\sqrt{\tau}}{2}\right\} \\ = \; & e^{-r\tau} K\left\{-\frac{\ln \frac{a}{S(t)} - r\tau}{\sigma^2\sqrt{\tau}} + \frac{\sqrt{\tau}}{2}\right\}\varphi\left(\frac{\ln \frac{a}{S(t)} - (r - \frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}}\right). \end{aligned}$$

3. Suppose $Z = (Z_1(t), Z_2(t))_{t \geq 0}$ is a standard Brownian motion in the plane. Find

$$E\left[\mid Z_1(t) - Z_2(t) \mid e^{(Z_1(t)+Z_2(t))^2}\right] \text{ if } 0 \leq t < \frac{1}{4}.$$

Solution. Note that $(Z_1(t)+Z_2(t), Z_1(t)-Z_2(t))$ is a Gaussian random vector in the plane such that $Z_1(t) \pm Z_2(t) \in N(0, 2t)$. Moreover, since

$$\mathrm{Cov}(Z_1(t) + Z_2(t), Z_1(t) - Z_2(t)) = E\left[(Z_1(t) + Z_2(t))(Z_1(t) - Z_2(t))\right]$$

$$= E\left[(Z_1^2(t) - Z_2^2(t))\right] = t - t = 0.$$

the random variables $Z_1(t)+Z_2(t)$ and $Z_1(t)-Z_2(t)$ are independent. Hence, if $0 \leq t < \frac{1}{4}$ and $G \in N(0, 1)$,

$$E\left[\mid Z_1(t) - Z_2(t) \mid e^{(Z_1(t)+Z_2(t))^2}\right]$$

$$= E\left[\mid Z_1(t) - Z_2(t) \mid\right] E\left[e^{(Z_1(t)+Z_2(t))^2}\right]$$

$$= E\left[\mid \sqrt{2t}G \mid\right] E\left[e^{(\sqrt{2t}G)^2}\right] = \sqrt{2t}\int_{\mathbf{R}} \mid x \mid e^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}} \int_{\mathbf{R}} e^{2tx^2} e^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}}$$

$$= 2\sqrt{2t}\int_0^\infty x e^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}} \int_{\mathbf{R}} e^{-\frac{(1-4t)x^2}{2}} \frac{dx}{\sqrt{2\pi}}$$

$$= 2\sqrt{\frac{t}{\pi}} \frac{1}{\sqrt{1-4t}} \int_{\mathbf{R}} e^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}} = 2\sqrt{\frac{t}{\pi(1-4t)}}.$$

4. Let $W = (W(t))_{t\geq 0}$ be a standard Brownian motion. (a) Prove that $W(s) - W(t) \in N(0, \mid s - t \mid)$. (b) Suppose a is a strictly positive real number and set $X = (\frac{1}{\sqrt{a}} W(at))_{t\geq 0}$. Prove that X is a standard Brownian motion.

5. (Black-Scholes model) A simple derivative of European type with the payoff $Y = g(S(T))$ at time of maturity T has the price $v(t, S(t))$ at time $t < T$, where

$$v(t, s) = e^{-r\tau} E\left[g(se^{(r-\frac{\sigma^2}{2})\tau+\sigma W(\tau)})\right]$$

and $\tau = T - t$. Use this formula to find the price of a European styled call with strike price K and time of maturity T.