SOLUTIONS
OPTIONS AND MATHEMATICS
(CTH[$mve095$], GU[$MMA700$])

May 24, 2010, morning (4 hours), v
No aids.
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Each problem is worth 3 points.

1. (Binomial model with 2 periods and $u > r > d$) A European derivative pays the amount Y at time of maturity $T = 2$, where

$$Y = \begin{cases} 0, & \text{if } X_1 = X_2 \\ 1, & \text{otherwise.} \end{cases}$$

(a) Find the price $\Pi_Y(0)$ of the derivative at time zero. (b) Suppose $(h_S(t), h_B(t))_{t=0}^T$ is a self-financing portfolio which replicates the derivative. Find $h_S(0)$.

Solution. (a) Set $v(t) = \Pi_Y(t)$ and

$$q_u = \frac{e^r - e^d}{e^u - e^r} = 1 - q_d.$$

Then

$$\begin{cases} v(2)_{|X_1=u,\ X_2=u} = 0 \\ v(2)_{|X_1=u,\ X_2=d} = 1 \\ v(2)_{|X_1=d,\ X_2=u} = 1 \\ v(2)_{|X_1=u,\ X_2=u} = 0 \end{cases}$$

and

$$\begin{cases} v(1)_{|X_1=u} = e^{-r}(q_u \cdot 0 + q_d \cdot 1) = e^{-r} q_d \\ v(1)_{|X_1=d} = e^{-r}(q_u \cdot 1 + q_d \cdot 0) = e^{-r} q_u. \end{cases}$$

Hence

$$\Pi_Y(0) = v(0) = e^{-r}(q_u e^{-r} q_d + q_d e^{-r} q_u) = 2e^{-2r} q_u q_d.$$

(b) We have that $h_S(0) = h_S(1)$ and $h_B(0) = h_B(1)$. Hence

$$\begin{cases} h_S(0)S(0)e^u + h_B(0)B(0)e^r = v(1)_{|X_1=u} \\ h_S(0)S(0)e^d + h_B(0)B(0)e^r = v(1)_{|X_1=d} \end{cases}$$

or

$$\begin{cases} h_S(0)S(0)e^u + h_B(0)B(0)e^r = e^{-r}q_d \\ h_S(0)S(0)e^d + h_B(0)B(0)e^r = e^{-r}q_u \end{cases}$$

and it follows that

$$h_S(0) = e^{-r}\frac{q_d - q_u}{S(0)(e^u - e^d)}.$$

2. (In this problem give only answers.) Let $Z(t) = (Z_1(t), Z_2(t))$, $t \geq 0$, be a standard Brownian motion in the plane and suppose $T > 0$. Set $U = e^{2Z_1(T)}$ and $V = e^{Z_1(T)+Z_2(2T)}$.

(a) Find $E[U]$, $E[V]$, $\mathrm{Var}(U)$, $\mathrm{Var}(V)$, and $\mathrm{Cov}(U,V)$. (b) Find an $a \in \mathbf{R}$ such that $\mathrm{Var}(U - aV) \leq \mathrm{Var}(U - xV)$ for every $x \in \mathbf{R}$?

Solution (to help the understanding of the answers). (a) In the following we will use that

$$a_1Z_1(t_1) + a_2Z_2(t_2) \in N(0, a_1^2t_1 + a_2^2t_2)$$

for all $a_1, a_2 \in \mathbf{R}$ and $t_1, t_2 \geq 0$. Hence, if $G \in N(0,1)$,

$$E[U] = E\left[e^{2\sqrt{T}G}\right] = e^{2T},$$
$$E[V] = E\left[e^{\sqrt{3T}G}\right] = e^{\frac{3}{2}T},$$
$$\mathrm{Var}(U) = E[U^2] - (E[U])^2 = E\left[e^{4\sqrt{T}G}\right] - e^{4T} = e^{8T} - e^{4T},$$
$$\mathrm{Var}(V) = E[V^2] - (E[V])^2 = E\left[e^{2\sqrt{3T}G}\right] - e^{3T} = e^{6T} - e^{3T},$$
$$\mathrm{Cov}(U,V) = E[UV] - E[U]E[V] = E\left[e^{\sqrt{11T}G}\right] - e^{2T}e^{\frac{3}{2}T} = e^{\frac{11}{2}T} - e^{\frac{7}{2}T}.$$

(b) Set $U_0 = U - E[U]$ and $V_0 = V - E[V]$. We have

$$f(x) =_{def} \mathrm{Var}(U - xV) = E\left[(U_0 - xV_0)^2\right]$$
$$= E\left[{U_0}^2\right] - 2xE[U_0V_0] + x^2E\left[{V_0}^2\right]$$
$$= (x\sqrt{E[{V_0}^2]} - \frac{E[U_0V_0]}{\sqrt{E[{V_0}^2]}})^2 + E\left[{U_0}^2\right] - (\frac{E[U_0V_0]}{\sqrt{E[{V_0}^2]}})^2.$$

Hence

$$\min f = f(a)$$

where

$$a = \frac{\mathrm{Cov}(U,V)}{\mathrm{Var}(V)} = \frac{e^{\frac{11}{2}T} - e^{\frac{7}{2}T}}{e^{6T} - e^{3T}}$$

$$= \frac{e^{\frac{5}{2}T} - e^{\frac{1}{2}T}}{e^{3T} - 1} = \frac{e^{\frac{1}{2}T}(e^T + 1)}{e^{2T} + e^T + 1}.$$

3. (Black-Scholes model) Suppose $0 < t_0 < T$ and $K > 0$. A financial derivative of European type pays the amount $Y = (\frac{S(T)}{S(t_0)} - K)^+$ at time of maturity T. Find the delta of the option at time t if (a) $0 < t < t_0$ (b) $t_0 < t < T$.

Solution. We first solve Part (b). Note that

$$Y = \frac{1}{S(t_0)}(S(T) - KS(t_0))^+$$

and, accordingly from this, if $t_0 \leq t < T$,

$$\Pi_Y(t) = \frac{1}{S(t_0)} c(t, S(t), KS(t_0), T)$$

$$= \frac{1}{S(t_0)} \left\{ S(t)\Phi(d_1(t)) - KS(t_0)e^{-r(T-t)}\Phi(d_2(t)) \right\}$$

where

$$d_1(t) = \frac{\ln \frac{S(t)}{KS(t_0)} + (r + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}$$

and

$$d_2(t) = \frac{\ln \frac{S(t)}{KS(t_0)} + (r - \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}.$$

In particular,

$$\Pi_Y(t_0)$$

$$= \Phi(\frac{-\ln K + (r + \frac{\sigma^2}{2})(T-t_0)}{\sigma\sqrt{T-t_0}}) - Ke^{-r(T-t_0)}\Phi(\frac{-\ln K + (r - \frac{\sigma^2}{2})(T-t_0)}{\sigma\sqrt{T-t_0}})$$

and, moreover, from the known delta of a European call we get

$$\Delta(t) = \frac{1}{S(t_0)}\Phi(\frac{\ln \frac{S(t)}{KS(t_0)} + (r + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}), \text{ if } t_0 < t < T.$$

We next treat Part (a). If $s = S(t)$ and $0 < t < t_0$,

$$\Pi_Y(t) = e^{-r(t_0-t)}\Pi_Y(t_0)$$

since $\Pi_Y(t_0)$ is known at time t. Moreover, $\Pi_Y(t)$ is independent of s and we have

$$\Delta(t) = 0, \text{ if } 0 < t < t_0.$$

4. (Dominance principle) State and prove the Put-Call Parity Theorem.

5. (Black-Scholes model) Consider a European call option on S with strike price K and time of maturity T. Prove that the delta of the call at time $t < T$ equals

$$\Phi(\frac{\ln \frac{S(t)}{K} + (r + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}).$$