SOLUTIONS
OPTIONS AND MATHEMATICS
(CTH[mve095], GU[MMA700])

August 28, 2010, morning V.
No aids.
Each problem is worth 3 points.
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1. Let X be a random variable with strictly positive variance and suppose a, b, c, and d are real numbers such that $bd \neq 0$. Show that

$$\mathrm{Cor}(a + bX, c + dX) = \frac{bd}{\mid bd \mid}.$$

Solution. Set

$$U = (a + bX) - E\left[a + bX\right] = b(X - E\left[X\right])$$

and

$$V = (c + dX) - E\left[c + dX\right] = d(X - E\left[X\right]).$$

Now

$$\mathrm{Cor}(a + bX, c + dX) = \frac{\mathrm{Cov}(a + bX, c + dX)}{\sqrt{\mathrm{Var}(a + bX)}\sqrt{\mathrm{Var}(c + dX)}}$$

$$= \frac{E\left[UV\right]}{\sqrt{E\left[U^2\right]}\sqrt{E\left[V^2\right]}} = \frac{bdE\left[(X - E\left[X\right])^2\right]}{\mid b \mid\mid d \mid E\left[(X - E\left[X\right])^2\right]} = \frac{bd}{\mid bd \mid}.$$

2. (Black-Scholes model) Suppose $K > 0$ and $0 = t_0 < t_1 < ... < t_n = T$. A financial derivative of European type pays the amount Y at time of maturity T, where

$$Y = \sum_{i=1}^{n} (S(t_i) - KS(t_{i-1}))^+.$$

Find $\Pi_Y(0)$.

Solution. Set

$$Y_i = (S(t_i) - KS(t_{i-1}))^+ \ , \ i = 1, ..., n$$

and note that

$$\Pi_Y(0) = \Pi_{\Sigma_1^n Y_i}(0) = \sum_{i=1}^{n} \Pi_{Yi}(0).$$

Moreover, for each $i \in \{1, ..., n\}$,

$$\Pi_{Yi}(t_i) = e^{-r(T-t_i)}(S(t_i) - KS(t_{i-1}))^+.$$

Now, if $G \in N(0,1)$, we have by the Black-Scholes call price formula

$$\Pi_{Yi}(t_{i-1}) = e^{-r(T-t_i)}\left\{S(t_{i-1})\Phi(d_1(i)) - KS(t_{i-1})e^{-r(t_i-t_{i-1})}\Phi(d_2(i))\right\}$$
$$= e^{-r(T-t_i)}S(t_{i-1})(\Phi(d_1(i)) - Ke^{-r(t_i-t_{i-1})}\Phi(d_2(i)))$$

where

$$d_1(i) = \frac{1}{\sigma\sqrt{t_i - t_{i-1}}}(\ln\frac{1}{K} + (r + \frac{\sigma^2}{2})(t_i - t_{i-1}))$$

and

$$d_2(i) = \frac{1}{\sigma\sqrt{t_i - t_{i-1}}}(\ln\frac{1}{K} + (r - \frac{\sigma^2}{2})(t_i - t_{i-1})).$$

Accordingly from this

$$\Pi_{Yi}(0) = S(0)e^{-r(T-t_i)}(\Phi(d_1(i)) - Ke^{-r(t_i-t_{i-1})}\Phi(d_2(i)))$$

and

$$\Pi_Y(0) = S(0)\sum_{i=1}^{n} e^{-r(T-t_i)}(\Phi(d_1(i)) - Ke^{-r(t_i-t_{i-1})}\Phi(d_2(i))).$$

3. Let $T > 0$ and consider two stock price processes

$$\begin{cases} S_1(t) = S_1(0)e^{\alpha_1 t + \sigma_1 W_1(t)}, \ 0 \le t \le T \\ S_2(t) = S_2(0)e^{\alpha_2 t + \sigma_2 W_2(t)}, \ 0 \le t \le T \end{cases}$$

governed by a bivariate geometric Brownian motion with correlation parameter $\rho \in]-1, 1[$. A portfolio is long 1000 shares of the first stock and short $\frac{1000 S_1(0)}{S_2(0)}$ shares of the second stock. Consequently, the corresponding portfolio $\mathcal{A}$ is of value zero at time zero, that is $V_{\mathcal{A}}(0) = 0$. Find $P[V_{\mathcal{A}}(T) > 0]$, $E[V_{\mathcal{A}}(T)]$, and $E[(V_{\mathcal{A}}(T))^2]$.

Solution. We have

$$V_{\mathcal{A}}(T)=K(e^{\alpha_1 T+\sigma_1 W_1(T)}-e^{\alpha_2 T+\sigma_2 W_2(T)})$$

where $K=1000S_1(0)$. Hence

$$P\left[V_{\mathcal{A}}(T)>0\right]=P\left[e^{\alpha_1 T+\sigma_1 W_1(T)}>e^{\alpha_2 T+\sigma_2 W_2(T)}\right]$$

$$=P\left[\sigma_1 W_1(T)-\sigma_2 W_2(T)>(\alpha_2-\alpha_1)T\right].$$

Set $X_\pm=\sigma_1 W_1(T)\pm\sigma_2 W_2(T)\in N(0,\sigma_\pm^2 T)$, where

$$\sigma_\pm^2 T=_{def} E\left[(\sigma_1 W_1(T)\pm\sigma_2 W_2(T))^2\right]=(\sigma_1^2\pm 2\rho\sigma_1\sigma_2+\sigma_2^2)T.$$

Now

$$P\left[V_{\mathcal{A}}(T)>0\right]=\Phi(\frac{(\alpha_2-\alpha_1)\sqrt{T}}{\sqrt{\sigma_1^2-2\rho\sigma_1\sigma_2+\sigma_2^2}}).$$

Moreover, if $G\in N(0,1)$,

$$E\left[e^{\xi G}\right]=e^{\frac{\xi^2}{2}},\ \xi\in\mathbf{R}$$

and it follows that

$$E\left[V_{\mathcal{A}}(T)\right]=K(e^{(\alpha_1+\frac{1}{2}\sigma_1^2)T}-e^{(\alpha_2+\frac{1}{2}\sigma_2^2)T})$$

and

$$E\left[(V_{\mathcal{A}}(T))^2\right]$$

$$=K^2 E\left[e^{2\alpha_1 T+2\sigma_1 W_1(T)}-2e^{(\alpha_1+\alpha_2)T+\sigma_1 W_1(T)+\sigma_2 W_2(T)}+e^{2\alpha_2 T+2\sigma_2 W_2(T)}\right]$$

$$=K^2(e^{2(\alpha_1+\sigma_1^2)T}-2e^{(\alpha_1+\alpha_2+\frac{1}{2}\sigma_1^2+\rho\sigma_1\sigma_2+\frac{1}{2}\sigma_2^2)T}+e^{2(\alpha_2+\sigma_2^2)T}).$$

4. Let $W=(W(t))_{t\geq 0}$ be a standard Brownian motion. (a) Prove that $W(s)-W(t)\in N(0,\mid s-t\mid)$. (b) Suppose a is a strictly positive real number and set $X=(\frac{1}{\sqrt{a}}W(at))_{t\geq 0}$. Prove that X is a standard Brownian motion.

5. (Dominance Principle) Show that the European call price $c(t,S(t),K,T)$ is a convex function of K.