SOLUTIONS
OPTIONS AND MATHEMATICS
(CTH[$mve095$], GU[$MMA700$])

January 15, 2011, morning, v.
No aids.
Each problem is worth 3 points.
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1. Suppose W denotes a standard Brownian motion. Find

$$E\left[(W(t)+W^2(t))e^{W^2(t)}\right]$$

for every $t \in [0,1/2[$.

Solution. We have

$$E\left[(W(t)+W^2(t))e^{W^2(t)}\right] = \int_{\mathbf{R}} \left(\sqrt{t}x + (\sqrt{t}x)^2\right) e^{(\sqrt{t}x)^2} e^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}}$$

$$= t\int_{\mathbf{R}} x^2 e^{tx^2} e^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}} = t\int_{\mathbf{R}} x^2 e^{-\frac{1}{2}(1-2t)x^2} \frac{dx}{\sqrt{2\pi}}$$

$$= \{y = \sqrt{1-2t}x\} = \frac{t}{(1-2t)^{\frac{3}{2}}} \int_{\mathbf{R}} y^2 e^{-\frac{y^2}{2}} \frac{dy}{\sqrt{2\pi}}$$

$$= \frac{t}{(1-2t)^{\frac{3}{2}}}.$$

2. (Binomial model; T periods, $d < 0 < r < u$) A financial derivative of European type pays the amount Y at time of maturity T, where

$$Y = \begin{cases} 0 \text{ if } S(T-1) \leq S(T) \\ 1 \text{ if } S(T-1) > S(T). \end{cases}$$

Find a self-financing portfolio strategy $h = (h_S(t), h_B(t))_{t=0}^{T}$ which replicates Y.

Solution. Set $\Pi_Y(t) = v(t)$,

$$q_u = \frac{e^r - e^d}{e^u - e^d} \text{ and } q_d = \frac{e^u - e^r}{e^u - e^d}.$$

We have $v(T-1, X_1 = x_1, ..., X_{T-1} = x_{T-1}) = e^{-r}(q_u \cdot 0 + q_d \cdot 1) = e^{-r} q_d$ for all $x_1, ..., x_{T-1} \in \{u, d\}$. Hence

$$v(t) = e^{-(T-1-t)r} e^{-r} q_d = e^{-(T-t)r} q_d \text{ if } 0 \leq t \leq T-1.$$

Now

$$h_S(t; x_1, ..., x_{t-1}) = 0 \text{ and } h_B(t; x_1, ..., x_{t-1}) = \frac{e^{-Tr} q_d}{B(0)} \text{ if } 1 \leq t \leq T-1$$

and, as usual, $h(0) = h(1)$. Moreover,

$$\begin{cases} h_S(T; x_1, ..., x_{T-1}) S(T-1) e^u + h_B(T; x_1, ..., x_{T-1}) B(T-1) e^r = 0 \\ h_S(T; x_1, ..., x_{T-1}) S(T-1) e^d + h_B(T; x_1, ..., x_{T-1}) B(T-1) e^r = 1 \end{cases}$$

and we get

$$\begin{cases} h_S(T; x_1, ..., x_{T-1}) = -\frac{1}{S(T-1)(e^u - e^d)} \\ h_B(T; x_1, ..., x_{T-1}) = \frac{e^{u-r}}{B(T-1)(e^u - e^d)}. \end{cases}$$

3. (Black-Scholes model) Suppose $0 < a < K$. A financial derivative of European type has the payoff $Y = g(S(T))$ at time of maturity T, where $g(x) = \mid x - K \mid$ if $x \notin \,]K-a, K+a[$ and $g(x) = 0$ if $x \in \,]K-a, K+a[$. Find a hedging portfolio for the derivative.

Solution. If $A \subseteq \,]0, \infty[$,

$$1_A(x) = \begin{cases} 1 \text{ if } x \in A \\ 0 \text{ if } x \notin A. \end{cases}$$

Now

$$g(x) = (K - a - x)^+ + a 1_{]0, K-a]}(x) + (x - K - a)^+ + a 1_{[K+a, \infty[}(x)$$

$$= K - a - x + (x - K + a)^+ + a - a 1_{]K-a, \infty[}(x)$$

$$+(x-K-a)^+ + a1_{[K+a,\infty[}(x).$$

$$= K - x + (x-K+a)^+ - a1_{]K-a,\infty[}(x)$$

$$+(x-K-a)^+ + a1_{[K+a,\infty[}(x).$$

Hence, if $\tau = T - t > 0$, $s = S(t)$, and $G \in N(0,1)$,

$$v(t,s) =_{def} \Pi_Y(t) = Ke^{-r\tau} - s + c(t,s,K-a,T)$$

$$-ae^{-r\tau}E\left[1_{]K-a,\infty[}(se^{(r-\frac{\sigma^2}{2})\tau-\sigma\sqrt{\tau}G})\right] + c(t,s,K+a,T)$$

$$+ae^{-r\tau}E\left[1_{[K+a,\infty[}(se^{(r-\frac{\sigma^2}{2})\tau-\sigma\sqrt{\tau}G})\right]$$

$$= Ke^{-r\tau} - s + c(t,s,K-a,T) + c(t,s,K+a,T)$$

$$-ae^{-r\tau}\int_{x<\frac{\ln\frac{s}{K-a}+(r-\frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}}}\varphi(x)dx + ae^{-r\tau}\int_{x\leq\frac{\ln\frac{s}{K+a}+(r-\frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}}}\varphi(x)dx$$

$$= Ke^{-r\tau} - s + c(t,s,K-a,T) + c(t,s,K+a,T) - ae^{-r\tau}\Phi(\frac{\ln\frac{s}{K-a}+(r-\frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}})$$

$$+ae^{-r\tau}\Phi(\frac{\ln\frac{s}{K+a}+(r-\frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}}).$$

Recall that

$$c(t,s,K,T) = s\Phi(\frac{\ln\frac{s}{K}+(r+\frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}}) - Ke^{-r\tau}\Phi(\frac{\ln\frac{s}{K}+(r-\frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}})$$

and

$$\frac{\partial c}{\partial s} = \Phi(\frac{\ln\frac{s}{K}+(r+\frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}}).$$

Now

$$h_S(t) = \frac{\partial v}{\partial s} = -1 + \Phi(\frac{\ln\frac{s}{K-a}+(r+\frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}}) + \Phi(\frac{\ln\frac{s}{K+a}+(r+\frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}})$$

$$-\frac{ae^{-r\tau}}{s\sigma\sqrt{\tau}}\varphi(\frac{\ln\frac{s}{K-a}+(r-\frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}}) + \frac{ae^{-r\tau}}{s\sigma\sqrt{\tau}}\varphi(\frac{\ln\frac{s}{K+a}+(r-\frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}})$$

where, as said above, $s = S(t)$. Moreover,

$$h_B(t) = \frac{v(t,s) - h_S(t)S(t)}{B(t)}.$$

4. Let $(X_n)_{n=1}^{\infty}$ be an i.i.d. such that $P[X_1 = 1] = P[X_1 = -1] = \frac{1}{2}$ and set

$$Y_n = \frac{1}{\sqrt{n}}(X_1 + ... + X_n),\ n \in \mathbf{N}_+.$$

Prove that $Y_n \to G$, where $G \in N(0,1)$.

5. (Black-Scholes model) Suppose $t < T$ and $\tau = T - t$. A simple financial derivative of European type with the payoff function $g \in \mathcal{P}$ and time of maturity T has the price

$$\Pi_{g(S(T))}(t) = e^{-r\tau} E\left[g(se^{(r-\frac{\sigma^2}{2})\tau+\sigma\sqrt{\tau}G})\right]$$

at time t, where $s = S(t)$ is the stock price at time t and $G \in N(0,1)$.

(a) A European call has the strike price K and termination date T. Show that the price at time t equals $s\Phi(d_1) - Ke^{-r\tau}\Phi(d_2)$, where

$$d_1 = \frac{1}{\sigma\sqrt{\tau}}(\ln\frac{s}{K} + (r + \frac{\sigma^2}{2})\tau)$$

and $d_2 = d_1 - \sigma\sqrt{\tau}$.

(b) Derive the delta of the call in Part (a).