OPTIONER OCH MATEMATIK (CTH[$TMA155$],GU[$MAM690$])

LÖSNINGAR

Skrivningsdag, skrivtid, sal: 20 maj 2006, 4 timmar, v

Inga hjälpmedel.

Varje uppgift ger maximalt 3 poäng.

1. (The one period binomial model, where $d < r < u$) Consider a call with the payoff $Y = (S(1) - K)^+$ at the termination date 1, where $S(0)e^d < K < S(0)e^u$. (a) Find the call price $\Pi_Y(0)$ at time 0. (b) Prove that $e^{-r}Y > \Pi_Y(0)$ if and only if $S(1) = S(0)e^u$.

Solution: (a) Let $S(0) = s$ and $S(1) = se^X$, where $X = u$ or d. We have

$$\begin{aligned}\Pi_Y(0) &= e^{-r}(q_u(se^u - K)^+ + q_d(se^d - K)^+) \\ &= e^{-r}q_u(se^u - K)\end{aligned}$$

where

$$q_u = \frac{e^r - e^d}{e^u - e^d}.$$

$ANSWER: \Pi_Y(0) = e^{-r}q_u(se^u - K)$

(b) The event

$$\left[e^{-r}Y > \Pi_Y(0)\right] = \left[(S(1) - K)^+ > q_u(se^u - K)\right]$$

$$= [S(1) > K + q_u(se^u - K)] = [S(1) > (1 - q_u)K + q_u se^u]$$

$$= [X = u] = [S(1) = S(0)e^u]$$

which proves the assertion in Problem 1(b).

2. A random variable X has the density function $f(x) = \frac{x^2}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$, $x \in \mathbf{R}$. Find the characteristic function $c_X(\xi) = E\left[e^{i\xi X}\right]$, $\xi \in \mathbf{R}$.

Solution. By partial integration,

$$\sqrt{2\pi}E\left[e^{i\xi X}\right]=\int_{-\infty}^{\infty}e^{i\xi x}x^2e^{-\frac{x^2}{2}}dx$$

$$=\left[-e^{i\xi x}xe^{-\frac{x^2}{2}}\right]_{-\infty}^{\infty}-\int_{-\infty}^{\infty}(-i\xi e^{i\xi x}x-e^{i\xi x})e^{-\frac{x^2}{2}}dx$$

$$=\int_{-\infty}^{\infty}(i\xi e^{i\xi x}x+e^{i\xi x})e^{-\frac{x^2}{2}}dx=i\xi\int_{-\infty}^{\infty}e^{i\xi x}xe^{-\frac{x^2}{2}}dx+\sqrt{2\pi}e^{-\frac{\xi^2}{2}}$$

$$=i\xi\left\{\left[-e^{i\xi x}e^{-\frac{x^2}{2}}\right]_{-\infty}^{\infty}+i\xi\int_{-\infty}^{\infty}e^{i\xi x}e^{-\frac{x^2}{2}}dx\right\}+\sqrt{2\pi}e^{-\frac{\xi^2}{2}}$$

$$=(1-\xi^2)\sqrt{2\pi}e^{-\frac{\xi^2}{2}}.$$

$ANSWER: c_X(\xi)=(1-\xi^2)e^{-\frac{\xi^2}{2}}.$

3. (Black-Scholes model) Suppose $t_0 < t_* < T$ and consider a financial derivative of European type with payoff $Y = \mid S(T) - S(t_*) \mid$ at time of maturity T. Find the delta $\Delta(t)$ of the derivative at time t if
(a) $t \in \,]t_*, T[$. (Hint: Problem 5).
(b) $t \in \,]t_0, t_*[$.
(c) Finally, compute $\Delta(t_*-) - \Delta(t_*+)$.

Solution. (a) We have

$$Y=(S(T)-S(t_*))^+ + (S(t_*)-S(T))^+$$

and, hence, for all $t_* \le t < T$,

$$\Pi_Y(t)=c(t,S(t),S(t_*),T)+p(t,S(t),S(t_*),T).$$

Thus by put-call parity,

$$\Pi_Y(t)=2c(t,S(t),S(t_*),T)+S(t_*)e^{-r(T-t)}-S(t)\text{ if }t_*\le t<T$$

and we get

$$\Delta(t) = 2\Phi(\frac{1}{\sigma\sqrt{T-t}}(\ln\frac{S(t)}{S(t_*)} + (r+\frac{\sigma^2}{2})(T-t))) - 1 \text{ if } t_* < t < T$$

$\leftarrow ANSWER$

(b) Since

$$c(t_*, S(t_*), S(t_*), T) = aS(t_*)$$

where

$$a = \Phi(\frac{1}{\sigma}(r+\frac{\sigma^2}{2})\sqrt{T-t_*}) - e^{-r(T-t_*)}\Phi(\frac{1}{\sigma}(r-\frac{\sigma^2}{2})\sqrt{T-t_*})$$

we get

$$\Pi_Y(t_*) = (2a + e^{-r(T-t_*)} - 1)S(t_*).$$

Thus by the dominance principle for all $t_0 < t < t_*$,

$$\Pi_Y(t) = (2a + e^{-r(T-t_*)} - 1)S(t)$$

$$= (2a + e^{-r(T-t_*)} - 1)S(t)$$

and we have

$$\Delta(t) = 2a + e^{-r(T-t_*)} - 1$$

$\leftarrow ANSWER$

(c) From the above,

$$\Delta(t_*-) - \Delta(t_*+) = e^{-r(T-t_*)}(1 - 2\Phi(\frac{1}{\sigma}(r-\frac{\sigma^2}{2})\sqrt{T-t_*}))$$

$\leftarrow ANSWER$

4. Let $W = (W(t))_{t\geq 0}$ be a standard Brownian motion. (a) Prove that $W(s) - W(t) \in N(0, | s-t |)$. (b) Suppose a is a strictly positive real number and set $X = (\frac{1}{\sqrt{a}}W(at))_{t\geq 0}$. Prove that X is a standard Brownian motion.

5. (Black-Scholes model) Suppose $\tau = T - t > 0$ and

$$d_1 = \frac{1}{\sigma\sqrt{\tau}}(\ln\frac{s}{K} + (r + \frac{\sigma^2}{2})\tau).$$

Prove that

$$\frac{\partial c}{\partial s}(t, s, K, T) = \Phi(d_1)\ .$$

(Hint: $c(t, s, K, T) = s\Phi(d_1) - Ke^{-r\tau}\Phi(d_2)$, where $d_2 = d_1 - \sigma\sqrt{\tau}$)

OPTIONER OCH MATEMATIK (CTH[$TMA155$],GU[$MAM690$])

LÖSNINGAR
Skrivningsdag, tid, sal: 2 sep 2006, fm, v
Inga hjälpmedel.
Skrivtid: 4 timmar
Varje uppgift ger maximalt 3 poäng.

1. (The one period binomial model, where $d < 0 < r < u$) Consider a call with the payoff $Y = (S(1) - S(0))^+$ at the termination date 1. Find the replicating strategy of the derivative at time 0.

Solution: Let $S(0) = s$ and $S(1) = se^X$, where $X = u$ or d. If (h_S, h_B) denotes the replicating strategy at time 0 we have

$$h_S se^u + h_B B(0)e^r = s(e^u - 1)$$

and

$$h_S se^d + h_B B(0)e^r = 0.$$

From this it follows that

$$h_S s(e^u - e^d) = s(e^u - 1)$$

and

$$h_S = \frac{e^u - 1}{e^u - e^d}.$$

Moreover, we get

$$h_B = -\frac{1}{B(0)} h_S s e^{d-r} = \frac{s e^{d-r}}{B(0)} \frac{1 - e^u}{e^u - e^d}.$$

$ANSWER:$ $\frac{e^u-1}{e^u-e^d}$ units of the stock and $\frac{se^{d-r}}{B(0)}\frac{1-e^u}{e^u-e^d}$ units of the bond.

2. (Black-Scholes model) Suppose $0 < t < T$ and consider a financial derivative of European type with payoff $Y = (S(T) - S(0))^2/S(T)$ at time of maturity T. Find the price $\Pi_Y(t)$ and the delta $\Delta(t)$ of the derivative at time t.

Solution. We have

$$Y = S(T) - 2S(0) + S(0)^2 S(T)^{-1}.$$

Here

$$\Pi_{S(T)}(t) = S(t)$$

and

$$\Pi_{S(0)}(t) = S(0)e^{-r\tau}$$

where $\tau = T - t$. Moreover,

$$\Pi_{S(T)^{-1}}(t) = e^{-r\tau} \int_{-\infty}^{\infty} (S(t) e^{(r-\frac{\sigma^2}{2})\tau + \sigma\sqrt{\tau}x})^{-1} e^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}}$$

$$= S(t)^{-1} e^{-r\tau} e^{-(r-\frac{\sigma^2}{2})\tau} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2} - \sigma\sqrt{\tau}x} \frac{dx}{\sqrt{2\pi}}$$

$$= S(t)^{-1} e^{(\sigma^2 - 2r)\tau} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x+\sigma\sqrt{\tau})^2} \frac{dx}{\sqrt{2\pi}} = S(t)^{-1} e^{(\sigma^2-2r)\tau}.$$

Hence

$$\Pi_Y(t) = S(t) - 2S(0)e^{-r\tau} + S(0)^2 e^{(\sigma^2-2r)\tau} S(t)^{-1} \leftarrow ANSWER$$

Now, if $\Pi_Y(t) = v(t, S(t))$,

$$\Delta(t) = \frac{\partial v}{\partial s}(t, S(t)) = 1 - S(0)^2 e^{(\sigma^2 - 2r)\tau} S(t)^{-2} ANSWER$$

3. Suppose X is a non-negative random variable with probability density f and such that $0 < E\left[X^2\right] < \infty$. Let $\mu = E\left[X\right]$ and suppose $\alpha \in \left[0, 1\right]$.
(a) Prove that

$$\int_{\alpha\mu}^{\infty} x f(x) dx \geq (1 - \alpha)\mu.$$

(b) Prove that

$$\int_{\alpha\mu}^{\infty} f(x) dx \geq (1 - \alpha)^2 \frac{(E\left[X\right])^2}{E\left[X^2\right]}.$$

Solution. (a) We have

$$0 \leq \mu = \int_0^{\alpha\mu} x f(x) dx + \int_{\alpha\mu}^{\infty} x f(x) dx$$

$$\leq \int_0^{\alpha\mu} \alpha\mu f(x) dx + \int_{\alpha\mu}^{\infty} x f(x) dx \leq \int_0^{\infty} \alpha\mu f(x) dx + \int_{\alpha\mu}^{\infty} x f(x) dx$$

$$= \alpha\mu + \int_{\alpha\mu}^{\infty} x f(x) dx$$

and, consequently,

$$\int_{\alpha\mu}^{\infty} x f(x) dx \geq (1 - \alpha)\mu.$$

(b) We have

$$\int_{\alpha\mu}^{\infty} x f(x) dx = \int_0^{\infty} x \sqrt{f(x)} 1_{[\alpha\mu, \infty[}(x) \sqrt{f(x)} dx$$

and the Cauchy-Schwarz inequality yields

$$\int_{\alpha\mu}^{\infty} x f(x) dx \leq (\int_0^{\infty} x^2 f(x) dx)^{\frac{1}{2}} (\int_0^{\infty} 1_{[\alpha\mu, \infty[}(x) f(x) dx)^{\frac{1}{2}}$$

$$= (\int_0^{\infty} x^2 f(x) dx)^{\frac{1}{2}} (\int_{\alpha\mu}^{\infty} f(x) dx)^{\frac{1}{2}}$$

and, hence,

$$\int_{\alpha\mu}^{\infty} f(x)dx \geq \frac{(\int_{\alpha\mu}^{\infty} xf(x)dx)^2}{\int_0^{\infty} x^2 f(x)dx}$$

$$\geq \frac{(1-\alpha)^2\mu^2}{\int_{-\infty}^{\infty} x^2 f(x)dx} = (1-\alpha)^2 \frac{(E[X])^2}{E[X^2]}.$$

4. (Dominance Principle) Show that the map

$$K \to c(t, S(t), K, T), \ K > 0$$

is convex.

5. Let $(X_n)_{n=1}^{\infty}$ be an i.i.d. such that $P[X_1 = 1] = P[X_1 = -1] = \frac{1}{2}$ and set

$$Y_n = \frac{1}{\sqrt{n}}(X_1 + ... + X_n), \ n \in \mathbf{N}_+.$$

Prove that $Y_n \to G$, where $G \in N(0,1)$.

OPTIONS AND MATHEMATICS

(CTH[$TMA155$], GU[$MAM690$])

January 20, 2007, morning (4 hours), v
No aids.
Examiner: Christer Borell, telephone number 0704 063 461
Each problem is worth 3 points.

Solutions

1. (The one period binomial model, where $d < 0 < r < u$) Suppose

$$S(0)e^d < K < S(0)e^u$$

and consider a put of European type with the payoff $Y = (K - S(1))^+$ at the termination date 1. Find the replicating strategy of the derivative at time 0.

Solution: Let $S(0) = s$ and $S(1) = se^X$, where $X = u$ or d. If (h_S, h_B) denotes the replicating strategy at time 0 we have

$$h_S se^u + h_B B(0)e^r = 0$$

and

$$h_S se^d + h_B B(0)e^r = K - se^d.$$

From this it follows that

$$h_S s(e^u - e^d) = se^d - K$$

and

$$h_S = \frac{1}{s}\frac{se^d - K}{e^u - e^d}.$$

Moreover, we get

$$h_B = -\frac{1}{B(0)} h_S se^{u-r} = \frac{e^{u-r}}{B(0)}\frac{K - se^d}{e^u - e^d}.$$

$ANSWER:$ $\frac{1}{S(0)}\frac{se^d-K}{e^u-e^d}$ units of the stock and $\frac{e^{u-r}}{B(0)}\frac{K-se^d}{e^u-e^d}$ units of the bond.

2. (Black-Scholes model) Suppose $0 < t < T$ and consider a financial derivative of European type with payoff

$$Y = \begin{cases} 1 \text{ if } S(T) > K \\ 0 \text{ if } S(T) \leq K \end{cases}$$

at time of maturity T. Find the price $\Pi_Y(t)$ and the delta $\Delta(t)$ of the derivative at time t. For which value of the stock price $S(t)$ is $\Delta(t)$ maximal?

Solution. We have

$$Y = g(S(T))$$

where

$$g(x) = \begin{cases} 1 & \text{if } x > K \\ 0 & \text{if } x \le K. \end{cases}$$

Thus, if $s = S(T)$ and $\tau = T - t$,

$$\Pi_Y(t) = e^{-r\tau} \int_{-\infty}^{\infty} g(se^{(r-\frac{\sigma^2}{2})\tau + \sigma\sqrt{\tau}x}) e^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}}$$

$$= e^{-r\tau} \int_{-d_2}^{\infty} e^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}} = e^{-r\tau}\Phi(d_2)$$

where

$$d_2 = \frac{1}{\sigma\sqrt{\tau}}(\ln \frac{s}{K} + (r - \frac{\sigma^2}{2})\tau).$$

From this we get

$$\Delta(t) = \frac{\partial}{\partial s}\Pi_Y(t) = \frac{e^{-r\tau}}{s\sigma\sqrt{2\pi\tau}} e^{-\frac{d_2^2}{2}}$$

and

$$\frac{\partial}{\partial s}\Delta(t) = -\frac{1}{s^2}\frac{e^{-r\tau}}{\sigma\sqrt{2\pi\tau}} e^{-\frac{d_2^2}{2}} \left(1 + \frac{1}{\sigma^2\tau}(\ln \frac{s}{K} + (r - \frac{\sigma^2}{2})\tau)\right).$$

Thus $\frac{\partial}{\partial s}\Delta(t) = 0$ if $s = s_*$, where

$$s_* = Ke^{-(r+\frac{\sigma^2}{2})\tau}.$$

Moreover, $\frac{\partial}{\partial s}\Delta(t) > 0$ if $s < s_*$ and $\frac{\partial}{\partial s}\Delta(t) < 0$ if $s > s_*$ and it follows that the delta of the option has a maximum for $S(t) = s_*$.

3. Set $X(t) = W(t) - tW(1)$ and $Y(t) = X(1-t)$ if $0 \le t \le 1$. Prove that the processes $(X(t))_{0\le t\le 1}$ and $(Y(t))_{0\le t\le 1}$ are equivalent in distribution.

Solution. Given $t_1, ..., t_n \in [0,1]$ an arbitrary linear combination of $X(t_1), ..., X(t_n)$ is a linear combination of $W(t_1), ..., W(t_n), W(1)$ and, hence a centred Gaussian random variable. In a similar way a linear combination of $Y(t_1), ..., Y(t_n)$ is a centred Gaussian random variable. Therefore it only remains to prove that the processes $(X(t))_{0\leq t\leq 1}$ and $(Y(t))_{0\leq t\leq 1}$ have the same covariance. To this end let $0 \leq s \leq t \leq 1$. Then

$$E\left[X(s)X(t)\right] = E\left[(W(s) - sW(1))(W(t) - tW(1)\right]$$

$$= E\left[W(s)W(t)\right] - tE\left[W(s)W(1)\right] - sE\left[W(1)W(t)\right] + stE\left[(W^2(1)\right]$$

$$= s - st - st + st = s - st$$

and

$$E\left[Y(s)Y(t)\right] = E\left[X(1-t)X(1-s)\right] = (1-t) - (1-t)(1-s) = s - st.$$

Thus $E\left[X(s)X(t)\right] = E\left[Y(s)Y(t)\right] = \min(s,t) - st$ for all $0 \leq s, t \leq 1$ and it follows that the processes $(X(t))_{0\leq t\leq 1}$ and $(Y(t))_{0\leq t\leq 1}$ are equivalent in distribution.

4. Suppose $a > 0$. Prove the Markov inequality

$$P\left[\mid X \mid \geq a\right] \leq \frac{1}{a} E\left[\mid X \mid\right].$$

5. (Black-Scholes model) Suppose $t < T$ and $\tau = T - t$. Prove that

$$c(t, s, K, T) = s\Phi(d_1) - Ke^{-r\tau}\Phi(d_2),$$

and

$$p(t, s, K, T) = Ke^{-r\tau}\Phi(-d_2) - s\Phi(-d_1)$$

where

$$d_1 = \frac{\ln \frac{s}{K} + (r + \frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}}$$

and

$$d_2 = \frac{\ln \frac{s}{K} + (r - \frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}}.$$

OPTIONS AND MATHEMATICS
(CTH[$mve095$], GU[$man690$])

May 26, 2007, morning (4 hours), v
No aids.
Examiner: Christer Borell, telephone number 0704 063 461
Each problem is worth 3 points.

Solutions

1. (The binomial model with $u > 0$, $d = -u$, $r = \frac{1}{2}u$, and $T = 2$). Suppose $g(x) = 1$ if $x = 0$ and $g(x) = 0$ if $x \neq 0$. A derivative of European type has the payoff $g(S(T) - S(0))$ at time of maturity T. (a) Find the price of the derivative at time 0. (b) Suppose the strategy h replicates the derivative. Find $h_S(0)$. The answers in Parts (a) and (b) may contain the martingale probabilities q_u and q_d.

Solution. (a) We have

$$q_u = \frac{e^r - e^d}{e^u - e^d} = \frac{e^{u/2} - e^{-u}}{e^u - e^{-u}}$$

and

$$q_d = 1 - q_u = \frac{e^u - e^{u/2}}{e^u - e^{-u}}.$$

Thus if $v(t)$ denotes the price of the derivative at time t,

$$\begin{aligned} v(2)_{|X_1=u,X_2=u} &= 0 \\ v(2)_{|X_1=u,X_2=d} &= 1 \\ v(2)_{|X_1=d,X_2=u} &= 1 \\ v(2)_{|X_1=d,X_2=d} &= 0 \end{aligned}$$

and

$$\begin{aligned} v(1)_{|X_1=u} &= e^{-r}(q_u 0 + q_d 1) = e^{-r} q_d \\ v(1)_{|X_1=d} &= e^{-r}(q_u 1 + q_d 0) = e^{-r} q_u. \end{aligned}$$

Now

$$v(0) = e^{-r}(q_u e^{-r} q_d + q_d e^{-r} q_u)$$

$$= 2e^{-2r} q_u q_d = 2e^{-u} q_u q_d.$$

(b) Recall that $h(0) = h(1)$ and

$$h_S(1)S(1) + h_B(1)B(1) = v(1)$$

or

$$\begin{aligned} h_S(1)S(0)e^u + h_B(1)B(0)e^r &= e^{-r} q_d \\ h_S(1)S(0)e^d + h_B(1)B(0)e^r &= e^{-r} q_{u.} \end{aligned}$$

Hence

$$h_S(0) = h_S(1) = e^{-u/2} \frac{1}{S(0)} \frac{q_d - q_u}{e^u - e^{-u}}$$

2. Suppose $Z = (Z_1(t), Z_2(t))_{t \geq 0}$ is a standard Brownian motion in the plane. Find

$$E\left[e^{|Z_1(t)+Z_2(t)|}\right] .$$

Solution. The process $X(t) = \frac{1}{\sqrt{2}} Z_1(t) + \frac{1}{\sqrt{2}} Z_2(t),\ t \geq 0$, is a standard Brownian motion since $(\frac{1}{\sqrt{2}})^2 + (\frac{1}{\sqrt{2}})^2 = 1$. Hence $X(t) \in N(0,t)$ and it follows that

$$E\left[e^{|Z_1(t)+Z_2(t)|}\right] = E\left[e^{\sqrt{2}|X(t)|}\right] = E\left[e^{\sqrt{2t}|G|}\right]$$

where $G \in N(0,1)$. Thus

$$E\left[e^{|Z_1(t)+Z_2(t)|}\right] = \int_{-\infty}^{\infty} e^{\sqrt{2t}|x| - \frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}} = 2\int_{-\infty}^{0} e^{-\sqrt{2t}x - \frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}}$$

$$= 2e^t \int_{-\infty}^{0} e^{-\frac{1}{2}(x+\sqrt{2t})^2} \frac{dx}{\sqrt{2\pi}} = 2e^t \int_{-\infty}^{\sqrt{2t}} e^{-\frac{1}{2}x^2} \frac{dx}{\sqrt{2\pi}} = 2e^t \Phi(\sqrt{2t}).$$

3. (Black-Scholes model) Suppose $0 < T_0 < T$ and consider a simple derivative of European type with the payoff $Y = \min(S(T_0), S(T))$ at time of maturity T. Find $\Pi_Y(t)$ for all $t \in [0, T_0]$.

Solution. If a and b are real numbers $\min(a,b)+\max(a,b)=a+b$ and, consequently, $\min(a,b)=a+b-\max(a,b)=b-\max(0,b-a)$. Therefore $Y=S(T)-\max(0,S(T)-S(T_0))$ and it follows that

$$\Pi_Y(T_0)=S(T_0)-c(T_0,S(T_0),S(T_0),T).$$

But $c(T_0,S(T_0),S(T_0),T)=S(T_0)c(T_0,1,1,T)$, where

$$c(T_0,1,1,T)=\Phi(\frac{r+\frac{\sigma^2}{2}}{\sigma}\sqrt{T-T_0})-e^{-r(T-T_0)}\Phi(\frac{r-\frac{\sigma^2}{2}}{\sigma}\sqrt{T-T_0}).$$

Hence, if we define $a=1-c(T_0,1,1,T)$,

$$\Pi_Y(T_0)=aS(T_0)$$

and it follows that

$$\Pi_Y(t)=aS(t) \text{ if } 0\le t\le T_0.$$

4. (Dominance Principle) Show that the European call price $c(t,S(t),K,T)$ is a convex function of K.

5. (Black-Scholes model) Assume $t,T\in\mathbf{R}$, $\tau=T-t>0$, and $g\in\mathcal{P}$.

(a) Define the price $\Pi_Y(t)$ at time t of a European derivative with payoff $g(S(T))$ at time of maturity T.

(b) Let

$$d_1=\frac{1}{\sigma\sqrt{\tau}}(\ln\frac{s}{K}+(r+\frac{\sigma^2}{2})\tau),$$

and $d_2=d_1-\sigma\sqrt{\tau}$. Show that

$$c(t,s,K,T)=s\Phi(d_1)-Ke^{-r\tau}\Phi(d_2).$$

SOLUTIONS: OPTIONS AND MATHEMATICS

(CTH[mve095], GU[man690])

September 1, 2007, morning (4 hours), v

No aids.

Examiner: Christer Borell, telephone number 0704 063 461
Each problem is worth 3 points.

1. (The binomial model in one period with $p_u = \frac{1}{2}$) Suppose $X = \ln \frac{S(1)}{S(0)}$. Show that

$$u = E[X] + \sqrt{\mathrm{Var}(X)}$$

and

$$d = E[X] - \sqrt{\mathrm{Var}(X)}\,.$$

Solution. We have

$$E[X] = \frac{1}{2}u + \frac{1}{2}d$$

and

$$E\left[X^2\right] = \frac{1}{2}u^2 + \frac{1}{2}d^2.$$

Consequently,

$$\mathrm{Var}(X) = \frac{1}{4}(u-d)^2$$

and it follows that

$$E[X] + \sqrt{\mathrm{Var}(X)} = \frac{1}{2}u + \frac{1}{2}d + \frac{1}{2}(u-d) = u$$

and

$$E[X] - \sqrt{\mathrm{Var}(X)} = \frac{1}{2}u + \frac{1}{2}d - \frac{1}{2}(u-d) = d.$$

2. (a) A random variable X has the density function

$$f(x) = \begin{cases} e^{-x}, \text{ if } x > 0, \\ 0, \text{ if } x \leq 0. \end{cases}$$

Find the characteristic function c_X of X (recall that $c_X(\xi) = E\left[e^{i\xi X}\right]$ if $\xi \in \mathbf{R}$).

(b) A random variable Y has the density function

$$g(x) = \begin{cases} 0, & \text{if } x > 0, \\ e^x, & \text{if } x \le 0. \end{cases}$$

Find the characteristic function c_Y of Y.

(c) A random variable Z has the density function $h(x) = \frac{1}{2}e^{-|x|}$, $x \in \mathbf{R}$. Find the characteristic function c_Z of Z.

Solution. (a) For each $\xi \in \mathbf{R}$,

$$c_X(\xi) = E\left[e^{i\xi X}\right] = \int_{-\infty}^{\infty} f(x)e^{i\xi x}dx = \int_0^{\infty} e^{-x}e^{i\xi x}dx$$

$$= \int_0^{\infty} e^{x(i\xi-1)}dx = \left[\frac{1}{i\xi - 1}e^{x(i\xi-1)}\right]_0^{\infty}.$$

Here $\mid e^{x(i\xi-1)} \mid = \mid e^{-x}e^{i\xi x} \mid = e^{-x} \mid e^{i\xi x} \mid = e^{-x}$ and we get

$$c_X(\xi) = \frac{1}{1 - i\xi}.$$

Alternatively, use that $e^{ia} = \cos a + i \sin a$ and compute

$$\int_0^{\infty} e^{-x}e^{i\xi x}dx = \int_0^{\infty} e^{-x} \cos \xi x dx + i \int_0^{\infty} e^{-x} \sin \xi x dx$$

by partial integration.

(b) Here $P\left[-Y \le y\right] = P\left[Y \ge -y\right] = \int_{-y}^{\infty} g(x)dx = \int_{-\infty}^{y} g(-t)dt = \int_{-\infty}^{y} f(t)dt$ and it follows that the random varibles $-Y$ and X have the same distribution. Consequently, $c_Y(\xi) = c_{-X}(\xi) = c_X(-\xi) = \frac{1}{1+i\xi}$.

(c) Since $h(x) = \frac{1}{2}f(x) + \frac{1}{2}g(x)$ and hence

$$c_Z(\xi) = \frac{1}{2}\int_{-\infty}^{\infty}(f(x) + g(x))e^{i\xi x}dx = \frac{1}{2}\left\{\frac{1}{1 - i\xi} + \frac{1}{1 + i\xi}\right\}$$

$$= \frac{1}{1+\xi^2}.$$

3. (Black-Scholes model) Suppose $0 < T_0 < t^* < T$ and $0 < \delta < 1$ and consider a simple derivative of European type with the payoff

$$Y = \mid S(T) - S(T_0) \mid$$

at time of maturity T. Find $\Pi_Y(0)$ if the stock pays the dividend $\delta S(T_0)$ at time t^*.

Solution. First note that

$$Y = 2(S(T) - S(T_0))^+ - S(T) + S(T_0).$$

If $s_0 = S(T_0)$ and

$$g(x) = 2(x - s_0)^+ - x + s_0$$

then

$$\Pi_Y(T_0) = \Pi_{g(S(T))}(T_0)$$

$$= e^{-r(T-T_0)} E\left[g((s_0 - \delta s_0 e^{-r(t^*-T_0)}) e^{(r-\frac{\sigma^2}{2})(T-T_0)+\sigma\sqrt{T-T_0}G}) \right]$$

$$= e^{-r(T-T_0)} E\left[g((s_0(1 - \delta e^{-r(t^*-T_0)}) e^{(r-\frac{\sigma^2}{2})(T-T_0)+\sigma\sqrt{T-T_0}G}) \right]$$

where $G \in N(0,1)$. Hence

$$\Pi_Y(T_0) = 2c(T_0, s_0(1 - \delta e^{-r(t^*-T_0)}), s_0, T) - s_0(1 - \delta e^{-r(t^*-T_0)}) + s_0 e^{-r(T-T_0)}$$

and we get

$$\Pi_Y(T_0) = S(T_0)\left\{(1 - \delta e^{-r(t^*-T_0)})A - e^{-r(T-T_0)}B - 1 + \delta e^{-r(t^*-T_0)} + e^{-r(T-T_0)}\right\}$$

where

$$A = 2\Phi(\frac{\ln(1 - \delta e^{-r(t^*-T_0)}) + (r + \frac{\sigma^2}{2})(T - T_0)}{\sigma\sqrt{T - T_0}})$$

and

$$B = 2\Phi(\frac{\ln(1 - \delta e^{-r(t^*-T_0)}) + (r - \frac{\sigma^2}{2})(T - T_0)}{\sigma\sqrt{T - T_0}}).$$

Since A and B are independent of $S(T_0)$ we conclude that

$$\Pi_Y(0) = S(0)\left\{(1-\delta e^{-r(t^*-T_0)})A - e^{-r(T-T_0)}B - 1 + \delta e^{-r(t^*-T_0)} + e^{-r(T-T_0)}\right\}.$$

4. Show that there exists an arbitrage portfolio in the binomial model in one period if and only if

$$r \notin \,]d,u[\,.$$

5. Let $W = (W(t))_{t\geq 0}$ be a standard Brownian motion. (a) Prove that $W(s) - W(t) \in N(0, \mid s-t \mid)$. (b) Suppose a is a strictly positive real number and set $X = (\frac{1}{\sqrt{a}}W(at))_{t\geq 0}$. Prove that X is a standard Brownian motion.

SOLUTIONS: OPTIONS AND MATHEMATICS

(CTH[mve095], GU[MAN690])

January 19, 2008, morning (4 hours), v
No aids.
Examiner: Christer Borell, telephone number 0705292322
Each problem is worth 3 points.

1. (Black-Scholes model) A derivative of European type pays the amount

$$Y = S(T) + \frac{1}{S(T)}$$

at time of maturity T. Find $\Pi_Y(t)$ for all $0 \leq t < T$.

Solution. We have

$$\Pi_Y(t) = \Pi_{S(T)}(t) + \Pi_{\frac{1}{S(T)}}(t).$$

Here, if $\tau = T-t$, $s = S(t)$, and $G \in N(0,1)$,

$$\Pi_{S(T)}(t) = e^{-r\tau}E\left[se^{(r-\frac{\sigma^2}{2})\tau+\sigma\sqrt{\tau}G}\right]$$

$$= se^{-\frac{\sigma^2}{2}\tau} E\left[e^{\sigma\sqrt{\tau}G}\right] = se^{-\frac{\sigma^2}{2}\tau} e^{\frac{\sigma^2}{2}\tau} = s.$$

Moreover,

$$\Pi_{\frac{1}{S(T)}}(t) = e^{-r\tau} E\left[\frac{1}{se^{(r-\frac{\sigma^2}{2})\tau+\sigma\sqrt{\tau}G}}\right]$$

$$= e^{-r\tau}\frac{e^{-(r-\frac{\sigma^2}{2})\tau}}{s} E\left[e^{-\sigma\sqrt{\tau}G}\right]$$

$$= \frac{e^{-(2r-\frac{\sigma^2}{2})\tau}}{s} e^{\frac{1}{2}\sigma^2\tau} = \frac{1}{s}e^{(\sigma^2-2r)\tau}$$

and it follows that

$$\Pi_Y(t) = S(t) + \frac{1}{S(t)}e^{(\sigma^2-2r)\tau}.$$

2. (Binomial model) Suppose $d = -u$ and $e^r = \frac{1}{2}(e^u + e^d)$. A financial derivative of European type has the maturity date $T = 4$ and payoff $Y = f(X_1 + X_2 + X_3 + X_4)$, where $f(x) = 1$ if $x \in \{4u, 0, -4u\}$ and $f(x) = -1$ if $x \in \{2u, -2u\}$. Show that $\Pi_Y(0) = 0$.

Solution. It follows that $d < r < u$ and

$$q_u = \frac{e^r - e^d}{e^u - e^d} = \frac{e^u - e^r}{e^u - e^d} = q_d.$$

Hence $q_u = q_d = \frac{1}{2}$. Furthemore,

$$\Pi_Y(0) = e^{-4r}\sum_{k=0}^{4}\left(\begin{array}{c}4\\k\end{array}\right) q_u^k q_d^{4-k} f(ku + (4-k)d)$$

$$= e^{-4r}\sum_{k=0}^{4}\left(\begin{array}{c}4\\k\end{array}\right) q_u^k q_d^{4-k} f((2k-4)u)$$

$$= e^{-4r}(\frac{1}{2})^4(1 - 4 + 6 - 4 + 1) = 0.$$

3. (Black-Scholes model) Suppose $T > 0$, $N \in \mathbf{N}_+$, $h = \frac{T}{N}$, and $t_n = nh$, $n = 0, ..., N$, and consider a derivative of European type paying the amount $Y = \sum_{n=0}^{N-1} (\ln \frac{S(t_{n+1})}{S(t_n)})^2$ at time of maturity T. Find $\Pi_Y(0)$.

Solution. First consider a derivative paying the amount $Y_n = (\ln \frac{S(t_{n+1})}{S(t_n)})^2$ at time T. Since Y_n is known at time t_{n+1}, $\Pi_{Y_n}(t_{n+1}) = Y_n e^{-r(T-t_{n+1})}$. Note that

$$S(t_{n+1}) = S(t_n) e^{(\mu - \frac{\sigma^2}{2})h + \sigma(W(t_{n+1}) - W(t_n))}$$

where $W(t_{n+1}) - W(t_n) \in N(0, h)$. Thus, if $G \in N(0, 1)$,

$$\Pi_{Y_n}(t_n) = e^{-rh} E\left[e^{-r(T-t_{n+1})} \left\{ (r - \frac{\sigma^2}{2})h + \sigma\sqrt{h}G \right\}^2 \right]$$

$$= e^{-r(T-t_n)} \left\{ (r - \frac{\sigma^2}{2})^2 h^2 + \sigma^2 h \right\}$$

and since the expression for $\Pi_{Y_n}(t_n)$ is known at time 0,

$$\Pi_{Y_n}(0) = e^{-t_n h} e^{-r(T-t_n)} \left\{ (r - \frac{\sigma^2}{2})^2 h^2 + \sigma^2 h \right\}$$

$$= e^{-rT} \left\{ (r - \frac{\sigma^2}{2})^2 h^2 + \sigma^2 h \right\}.$$

Now it follows that

$$\Pi_Y(0) = \sum_{n=0}^{N-1} \Pi_{Y_n}(0) = N e^{-rT} \left\{ (r - \frac{\sigma^2}{2})^2 h^2 + \sigma^2 h \right\}$$

$$= T e^{-r\tau} \left\{ \sigma^2 + h(r - \frac{\sigma^2}{2})^2 \right\}.$$

4. Derive the delta of a European call in the Black-Scholes model. Recall that the call price equals $s\Phi(d_1) - Ke^{-r\tau}\Phi(d_2)$, where $s = S(t)$, $\tau = T - t > 0$, and

$$d_1 = \frac{\ln \frac{s}{K} + (r + \frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}} = d_2 + \sigma\sqrt{\tau}.$$

5. Consider the binomial model in one period and assume $d < r < u$. A derivative pays the amount $Y = f(X)$ at time 1. Find a portfolio which replicates the derivative at time 0.

SOLUTIONS

OPTIONS AND MATHEMATICS

(CTH[$mve095$], GU[$MMA700$])

May 24, 2008, morning (4 hours), v
No aids.
Examiner: Torbjörn Lundh, telephone number 0731 526320
Each problem is worth 3 points.

1. (Binomial model) Suppose $T = 3$, $u > r > 0$, and $d = -u$. A derivative of European type has the payoff Y at time of maturity T, where

$$Y = \begin{cases} 1, & \text{if } X_1 = X_2 = X_3, \\ 0, & \text{otherwise.} \end{cases}$$

Find $\Pi_Y(0)$ (the answer may contain the martingale probabilities q_u and q_d, which must, however, be defined explicitely).

Solution. We have

$$q_u = \frac{e^r - e^{-u}}{e^u - e^{-u}} \text{ and } q_d = \frac{e^u - e^r}{e^u - e^{-u}}.$$

Introducing $\Pi_Y(t) = v(t)$, it follows that

$$\begin{cases} v(2)_{|X_1=u,X_2=u} = e^{-r}(q_u \cdot 1 + q_d \cdot 0) = e^{-r}q_u \\ v(2)_{|X_1=u,X_2=d} = e^{-r}(q_u \cdot 0 + q_d \cdot 0) = 0 \\ v(2)_{|X_1=d,X_2=u} = e^{-r}(q_u \cdot 0 + q_d \cdot 0) = 0 \\ v(2)_{|X_1=d,X_2=d} = e^{-r}(q_u \cdot 0 + q_d \cdot 1) = e^{-r}q_d \end{cases}$$

and

$$\begin{cases} v(1)_{|X_1=u} = e^{-r}(q_u e^{-r} q_u + q_d \cdot 0) = e^{-2r} q_u^2 \\ v(1)_{|X_1=d} = e^{-r}(q_u \cdot 0 + q_d e^{-r} q_d) = e^{-2r} q_d^2. \end{cases}$$

Thus

$$v(0) = e^{-r}(q_u e^{-2r} q_u^2 + q_d e^{-2r} q_d^2) = e^{-3r}(q_u^3 + q_d^3).$$

Alternative solution. We have $Y = 1_{\{S(0)e^{3u}, S(0)e^{-3u}\}}(S(3))$ and the derivative is simple. Hence

$$\Pi_Y(0) = e^{-3r} \sum_{k=0}^{3} \binom{3}{k} q_u^k q_d^{3-k} 1_{\{S(0)e^{3u}, S(0)e^{-3u}\}}(S(0)e^{ku+(3-k)(-u)})$$

$$= e^{-3r} \sum_{k \in \{0,3\}} \binom{3}{k} q_u^k q_d^{3-k} = e^{-3r}(q_u^3 + q_d^3).$$

2. Suppose $Z = (Z_1(t), Z_2(t))_{t \geq 0}$ is a standard Brownian motion in the plane and define $R(t) =\mid Z(t) \mid= \sqrt{Z_1^2(t) + Z_2^2(t)}$, $t \geq 0$. Find $E\left[e^{\xi R^2(t)}\right]$ if $t > 0$ and $\xi < \frac{1}{2t}$.

Solution. Suppose $t > 0$, $\xi < \frac{1}{2t}$, and $G \in N(0,1)$. Then

$$E\left[e^{\xi R^2(t)}\right] = E\left[e^{\xi Z_1^2(t)} e^{\xi Z_2^2(t)}\right] = E\left[e^{\xi Z_1^2(t)}\right] E\left[e^{\xi Z_2^2(t)}\right]$$

$$= (E\left[e^{\xi t G^2}\right])^2$$

and setting $\eta = \xi t$,

$$E\left[e^{\eta G^2}\right] = \int_{-\infty}^{\infty} e^{\eta x^2} e^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}}$$

$$= \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}(1-2\eta)} \frac{dx}{\sqrt{2\pi}} = \int_{-\infty}^{\infty} e^{-\frac{y^2}{2}} \frac{dy}{\sqrt{2\pi(1-2\eta)}}$$

$$= \frac{1}{\sqrt{1-2\eta}}.$$

Hence,

$$E\left[e^{\xi R^2(t)}\right] = \frac{1}{1-2\xi t}.$$

3. (Black-Scholes model) A derivative of European type pays the amount

$$Y = 1 + S(T)\ln S(T)$$

at time of maturity T. (a) Find $\Pi_Y(t)$. (b) Find a hedging portfolio of the derivative at time t.

Solution. (a) If $s = S(t)$, $\tau = T - t$, and $G \in N(0,1)$, then

$$\Pi_Y(t) = e^{-r\tau}E\left[1 + se^{(r-\frac{\sigma^2}{2})\tau+\sigma\sqrt{\tau}G}\left\{\ln s + (r - \frac{\sigma^2}{2})\tau + \sigma\sqrt{\tau}G\right\}\right]$$

$$= e^{-r\tau} + s\left\{\ln s + (r - \frac{\sigma^2}{2})\tau\right\}e^{-\frac{\sigma^2}{2}\tau}E\left[e^{\sigma\sqrt{\tau}G}\right] + s\sigma\sqrt{\tau}E\left[Ge^{-\frac{\sigma^2}{2}\tau+\sigma\sqrt{\tau}G}\right]$$

$$= e^{-r\tau} + s\left\{\ln s + (r - \frac{\sigma^2}{2})\tau\right\} + s\sigma\sqrt{\tau}\int_{-\infty}^{\infty} xe^{-\frac{\sigma^2}{2}\tau+\sigma\sqrt{\tau}x-\frac{x^2}{2}}\frac{dx}{\sqrt{2\pi}}$$

$$= e^{-r\tau} + s\left\{\ln s + (r - \frac{\sigma^2}{2})\tau\right\} + s\sigma\sqrt{\tau}\int_{-\infty}^{\infty} xe^{-\frac{(x-\sigma\sqrt{\tau})^2}{2}}\frac{dx}{\sqrt{2\pi}} =$$

$$= e^{-r\tau} + s\left\{\ln s + (r - \frac{\sigma^2}{2})\tau\right\} + s\sigma\sqrt{\tau}\int_{-\infty}^{\infty} (y+\sigma\sqrt{\tau})e^{-\frac{y^2}{2}}\frac{dx}{\sqrt{2\pi}}$$

$$= e^{-r\tau} + s\left\{\ln s + (r - \frac{\sigma^2}{2})\tau\right\} + s\sigma^2\tau$$

$$= e^{-r\tau} + s\ln s + s(r + \frac{\sigma^2}{2})\tau$$

$$= e^{-r\tau} + S(t)\ln S(t) + S(t)(r + \frac{\sigma^2}{2})\tau$$

(b) A portfolio with

$$h_S(t) = \left(\frac{\partial}{\partial s}\left\{e^{-r\tau} + s\ln s + s(r + \frac{\sigma^2}{2})\tau\right\}\right)_{|s=S(t)}$$

$$= 1 + (r + \frac{\sigma^2}{2})\tau + \ln S(t)$$

units of the stock and

$$h_B(t)$$

$$= (e^{-r\tau} + S(t)\ln S(t) + S(t)(r + \frac{\sigma^2}{2})\tau - S(t)(1 + (r + \frac{\sigma^2}{2})\tau + \ln S(t)))/B(t)$$

$$= (e^{-r\tau} - S(t))/B(t)$$

units of the bond is a hedging portfolio at time t.

4. (Dominance Principle) State and prove the Put-Call Parity relation.

5. (Dominance Principle) Suppose $t_0 < t^* < T$ and let D be a positive number, which is known at time t_0. Now consider an American put with strike K and time of maturity T, where the underlying stock pays the dividend D at time t^* and

$$D \geq K(e^{r(t^*-t_0)} - 1).$$

Prove that it is not optimal to exercise the put in the time interval $]t_0, t^*[$.

SOLUTIONS

OPTIONS AND MATHEMATICS

(CTH[mve095], GU[MMA700])

August 30, 2008, morning (4 hours), V
No aids.
Examiner: Christer Borell, telephone number 0705292322
Each problem is worth 3 points.

1. (Black-Scholes model) A derivative of European type pays the amount $Y = \frac{S(T)}{S(T/2)}$ at time of maturity T. Find $\Pi_Y(0)$.

Solution. For any $t \in [0,T]$ and real number a, $\Pi_{aS(T)}(t) = aS(t)$ and, hence,

$$\Pi_Y(T/2) = \Pi_{\frac{1}{S(T/2)}S(T)}(T/2) = \frac{1}{S(T/2)}\Pi_{S(T)}(T/2)$$

$$= \frac{1}{S(T/2)}S(T/2) = 1.$$

Accordingly from this,

$$\Pi_Y(0) = e^{-\frac{rT}{2}}.$$

2. Suppose $Z = (Z_1(t), Z_2(t))_{t\geq 0}$ is a standard Brownian motion in the plane. Find $E\left[\sqrt{Z_1^2(t) + Z_2^2(t)}\right]$ if $t \geq 0$.

Solution. Let $t \geq 0$ be fixed. Since $(Z_1(t), Z_2(t))$ has the same distribution as $\sqrt{t}(Z_1(1), Z_2(1))$,

$$E\left[\sqrt{Z_1^2(t) + Z_2^2(t)}\right] = E\left[\sqrt{t(Z_1^2(1) + Z_2^2(1))}\right]$$

$$= \sqrt{t}\iint_{\mathbf{R}^2} \sqrt{x^2+y^2}e^{-\frac{x^2+y^2}{2}}\frac{dxdy}{2\pi} = \left[\begin{array}{c}\text{polar}\\ \text{coordinates}\end{array}\right]$$

$$= \sqrt{t}\int_0^\infty \int_0^{2\pi} r^2 e^{-\frac{r^2}{2}}\frac{drd\theta}{2\pi} = \sqrt{t}\int_0^\infty r^2 e^{-\frac{r^2}{2}}dr = \left[\begin{array}{c}\text{partial}\\ \text{integration}\end{array}\right]$$

$$= \sqrt{t}\int_0^\infty e^{-\frac{r^2}{2}}dr = \sqrt{\frac{\pi t}{2}}.$$

3. (Black-Scholes model) Suppose K is a positive real number and consider a simple derivative of European type with the payoff

$$Y = (\frac{1}{S(T)} - K)^+$$

at time of maturity T. Moreover, suppose $0 < t^* < T$ and $0 < \delta < 1$. Find $\Pi_Y(0)$ if the stock pays the dividend $\delta S(t^*-)$ at time t^*.

Solution. Let $s = S(0)$ and suppose $G \in N(0,1)$. We have

$$\Pi_Y(0) = e^{-rT} E\left[\left(\frac{1}{(1-\delta)se^{(r-\frac{\sigma^2}{2})T+\sigma\sqrt{T}G}} - K \right)^+ \right]$$

$$= \frac{e^{-rT}}{(1-\delta)s} E\left[(e^{-(r-\frac{\sigma^2}{2})T-\sigma\sqrt{T}G} - L)^+ \right]$$

where $L = (1-\delta)sK$. Here

$$E\left[(e^{-(r-\frac{\sigma^2}{2})T-\sigma\sqrt{T}G} - L)^+ \right] = \int_{-\infty}^{-\frac{1}{\sigma\sqrt{T}}(\ln L+(r-\frac{\sigma^2}{2})T)} (e^{-(r-\frac{\sigma^2}{2})T-\sigma\sqrt{T}x} - L) e^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}}$$

$$= e^{(\sigma^2-r)T} \int_{-\infty}^{-\frac{1}{\sigma\sqrt{T}}(\ln L+(r-\frac{\sigma^2}{2})T)} e^{-\frac{1}{2}(x+\sigma\sqrt{T})^2} \frac{dx}{\sqrt{2\pi}} - L\Phi(-\frac{1}{\sigma\sqrt{T}}(\ln L+(r-\frac{\sigma^2}{2})T))$$

$$= e^{(\sigma^2-r)T}\Phi(-\frac{1}{\sigma\sqrt{T}}(\ln L + (r - \frac{3}{2}\sigma^2)T)) - L\Phi(-\frac{1}{\sigma\sqrt{T}}(\ln L + (r - \frac{\sigma^2}{2})T)).$$

Thus

$$\Pi_Y(0) = \frac{e^{(\sigma^2-2r)T}}{(1-\delta)s}\Phi(-\frac{1}{\sigma\sqrt{T}}(\ln L+(r-\frac{3}{2}\sigma^2)T)) - e^{-rT}K\Phi(-\frac{1}{\sigma\sqrt{T}}(\ln L+(r-\frac{\sigma^2}{2})T)).$$

4. Prove that there exists an arbitrage portfolio in the single-period binomial model if and only if

$$r \notin]d, u[\,.$$

5. (Black-Scholes model) Consider a European call on a stock with price process $(S(t))_{t\geq 0}$. If K denotes strike price and T time of maturity, the Black-Scholes price of the call at time $t < T$ equals

$$c(t, S(t), K, T)) = s\Phi(d_1) - Ke^{-r\tau}\Phi(d_2),$$

where $\tau = T - t$ and

$$d_1 = d_2 + \sigma\sqrt{\tau} = \frac{1}{\sigma\sqrt{\tau}}(\ln\frac{S(t)}{K} + (r + \frac{\sigma^2}{2})\tau).$$

(a) Find the delta of the call.
(b) How is the call price formula changed if the stock price pays the dividend D at time $t^* \in]t, T[$, where D is a fixed amount known at time t?

SOLUTIONS
OPTIONS AND MATHEMATICS
(CTH[mve095], GU[MMA700])

January 17, 2009, morning (4 hours), v
No aids.
Examiner: Christer Borell, telephone number 0705292322
Each problem is worth 3 points.

1. (The one period binomial model, where $d < 0 < r < u$) Consider a put with the payoff $Y = (S(0) - S(1))^+$ at the termination date 1. Find the replicating strategy of the derivative at time 0.

Solution: Let $S(0) = s$ and $S(1) = se^X$, where $X = u$ or d. If (h_S, h_B) denotes the replicating strategy at time 0 we have

$$h_S s e^u + h_B B(0) e^r = 0$$

and

$$h_S s e^d + h_B B(0) e^r = s(1 - e^d).$$

From this it follows that

$$h_S s(e^u - e^d) = s(e^d - 1)$$

and

$$h_S = \frac{e^d - 1}{e^u - e^d}.$$

Moreover, we get

$$h_B = -\frac{1}{B(0)} h_S s e^{u-r} = \frac{s e^{u-r}}{B(0)} \frac{1 - e^d}{e^u - e^d}.$$

2. Suppose $\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$ and $\Phi(x) = \int_{-\infty}^{x} \varphi(t)dt$, $-\infty < x < \infty$. Prove that

$$1 - \Phi(x) \leq \frac{\varphi(x)}{x}, \text{ if } x > 0,$$

and

$$1 - \Phi(x) \geq \frac{x\varphi(x)}{1 + x^2}, \text{ if } x \in \mathbf{R}.$$

Solution. For any $x > 0$,

$$1 - \Phi(x) = \int_{x}^{\infty} \varphi(t)dt = \int_{x}^{\infty} \frac{1}{t} t\varphi(t)dt$$

$$\leq \int_{x}^{\infty} \frac{1}{x} t\varphi(t)dt = \frac{1}{x} \left[-\varphi(t)\right]_{t=x}^{t=\infty} = \frac{\varphi(x)}{x}.$$

This proves the first inequality. To prove the second inequality define

$$f(x) = (1 + x^2)(1 - \Phi(x)) - x\varphi(x), \text{ if } x \in \mathbf{R}.$$

It is obivous that $f(x) > 0$ if $x \leq 0$ and therefore it is enough to prove that $f(x) \geq 0$ for every $x > 0$. To this end, first note that

$$\lim_{x \to \infty} (1 + x^2)(1 - \Phi(x)) = 0$$

since $0 \leq 1 - \Phi(x) \leq \frac{\varphi(x)}{x} = \frac{1}{x\sqrt{2\pi}} e^{-\frac{x^2}{2}}$ for every $x > 0$. Hence

$$\lim_{x \to \infty} f(x) = 0$$

and it is enough to show that $f'(x) \leq 0$ if $x > 0$. Now for every $x > 0$,

$$f'(x) = 2x(1 - \Phi(x)) - (1 + x^2)\varphi(x) - \varphi(x) + x^2\varphi(x)$$

$$= 2x(1 - \Phi(x) - \frac{\varphi(x)}{x}) \leq 0$$

and we are done.

3. (Black-Scholes model) (a) Consider a derivative of European type with the payoff

$$Y = \frac{1}{n} \sum_{k=1}^{n} S(\frac{kT}{n})$$

at time of maturity T. Find $\Pi_Y(0)$.

(b) Consider a derivative of European type with the payoff

$$Z = \left\{ \prod_{k=1}^{n} S(\frac{kT}{n}) \right\}^{\frac{1}{n}}$$

at time of maturity T. Find $\Pi_Z(0)$. (Hint: $1^2 + 2^2 + ... + n^2 = \frac{n(n+1)(2n+1)}{6}$)

Solution. (a) Consider a derivative paying the amount $Y_k = S(\frac{kT}{n})$ at time T. Then

$$\Pi_Y(0) = \frac{1}{n} \sum_{k=1}^{n} \Pi_{Y_k}(0).$$

Moreover, $\Pi_{Y_k}(\frac{kT}{n}) = e^{-(T-\frac{kT}{n})r} S(\frac{kT}{n})$ and, hence,

$$\Pi_{Y_k}(0) = e^{-(T-\frac{kT}{n})r} S(0).$$

Thus

$$\Pi_Y(0) = \frac{S(0)}{n} \sum_{k=1}^{n} e^{-(1-\frac{k}{n})Tr}$$

$$= \frac{S(0)}{n} \sum_{i=0}^{n-1} e^{-iTr/n} = \frac{S(0)}{n} \frac{1 - e^{-Tr}}{1 - e^{-Tr/n}}.$$

(b) If $S(0) = s$,

$$\Pi_Z(0) = e^{-rT} E\left[\left\{ \prod_{k=1}^{n} s e^{(r-\frac{\sigma^2}{2})\frac{kT}{n} + \sigma W(\frac{kT}{n})} \right\}^{\frac{1}{n}} \right]$$

$$= s e^{-rT + (r-\frac{\sigma^2}{2})\frac{(n+1)T}{2n}} E\left[e^{\frac{\sigma}{n} \sum_{k=1}^{n} W(\frac{kT}{n})} \right].$$

Set $V_i = W(\frac{iT}{n})$, $i = 0, ..., n$. Then

$$\sum_{k=1}^{n} W(\frac{kT}{n}) = V_1 + ... + V_n$$

$$= V_1 + ... + V_{n-2} + 2V_{n-1} + (V_n - V_{n-1})$$
$$= V_1 + ... + V_{n-3} + 3V_{n-2} + 2(V_{n-1} - V_{n-2}) + (V_n - V_{n-1})$$
$$= n(V_1 - V_0) + ... + 2(V_{n-1} - V_{n-2}) + (V_n - V_{n-1})$$

and we get

$$E\left[e^{\frac{\sigma}{n}\sum_{k=1}^{n} W(\frac{kT}{n})}\right] = \prod_{k=1}^{n} E\left[e^{\frac{\sigma(n+1-k)}{n}(V_k - V_{k-1})}\right] = e^{\frac{\sigma^2}{2n^2}(n^2+...+2^2+1^2)\frac{T}{n}}$$

$$= e^{\frac{\sigma^2}{2n^2}\frac{n(n+1)(2n+1)}{6}\frac{T}{n}} = e^{\sigma^2 T\frac{(n+1)(2n+1)}{12n^2}}.$$

Thus

$$\Pi_Z(0) = se^{-rT+(r-\frac{\sigma^2}{2})\frac{(n+1)T}{2n}+\sigma^2 T\frac{(n+1)(2n+1)}{12n^2}} = S(0)e^{(\frac{1-n}{2n}r+\frac{1-n^2}{12n^2}\sigma^2)T}.$$

4. Let $(X_n)_{n=1}^{\infty}$ be an i.i.d. such that $P[X_1 = 1] = P[X_1 = -1] = \frac{1}{2}$ and set

$$Y_n = \frac{1}{\sqrt{n}}(X_1 + ... + X_n),\ n \in \mathbf{N}_+.$$

Prove that $Y_n \to G$, where $G \in N(0,1)$.

5. (Black-Scholes model) Suppose $\tau = T - t > 0$ and

$$d_1 = \frac{1}{\sigma\sqrt{\tau}}(\ln\frac{s}{K} + (r + \frac{\sigma^2}{2})\tau).$$

Prove that

$$\frac{\partial c}{\partial s}(t, s, K, T) = \Phi(d_1).$$

(Hint: $c(t, s, K, T) = s\Phi(d_1) - Ke^{-r\tau}\Phi(d_2)$, where $d_2 = d_1 - \sigma\sqrt{\tau}$)

SOLUTIONS

OPTIONS AND MATHEMATICS
(CTH[$mve095$], GU[$MMA700$])

May 25, 2009, morning (4 hours), m
No aids.
Examiner: Christer Borell, telephone number 0705292322
Each problem is worth 3 points.

1. (The one period binomial model, where $0 < d < r < u$) Consider a call with the payoff $Y = \frac{1}{2} \mid \frac{S(1)}{S(0)} - \frac{S(0)}{S(1)} \mid$ at the termination date 1. Find the replicating strategy of the derivative at time 0.

Solution: Let $S(0) = s$ and $S(1) = se^X$, where $X = u$ or d. If (h_S, h_B) denotes the replicating strategy at time 0 we have

$$h_S se^u + h_B B(0)e^r = \sinh u$$

and

$$h_S se^d + h_B B(0)e^r = \sinh d.$$

From this it follows that

$$h_S s(e^u - e^d) = \sinh u - \sinh d$$

and

$$h_S = \frac{\sinh u - \sinh d}{s(e^u - e^d)}.$$

Moreover, we get

$$h_B B(0)(e^{r+u} - e^{r+d}) = e^u \sinh d - e^d \sinh u$$

and

$$h_B = \frac{e^u \sinh d - e^d \sinh u}{B(0)e^r(e^u - e^d)}.$$

$ANSWER:$ $\frac{\sinh u - \sinh d}{S(0)(e^u - e^d)}$ (or $= \frac{1}{2S(0)}(1+e^{-u-d})$) units of the stock and $\frac{e^u \sinh d - e^d \sinh u}{B(0)e^r(e^u - e^d)}$ (or $= -\frac{e^{-r}}{2B(0)}(e^{-u} + e^{-d})$) units of the bond.

2. (Black-Scholes model) Suppose t^*, T_0, T, and δ are positive numbers satisfying the inequalities $T_0 < t^* < T$ and $\delta < 1$. Moreover, suppose $t < t^*$. A stock pays the dividend $\delta S(t^*-)$ at time t^*. Find the price $\Pi_Y(t)$ at time t of a derivative of European type paying the amount

$$Y = (\frac{S(T)}{S(T_0)} - 1)^+$$

at time of maturity T.

Solution. Set $s = S(t)$ and $\tau = T - t$.

To begin with we assume $T_0 \le t < t^*$. If

$$g(x) = (\frac{x}{S(T_0)} - 1)^+ = \frac{1}{S(T_0)}(x - S(T_0))^+$$

we know that

$$\Pi_Y(t) = e^{-r\tau} E\left[g((1-\delta)se^{(r-\frac{\sigma^2}{2})\tau + \sigma\sqrt{\tau}G})\right]$$

where $G \in N(0,1)$. Hence, by the Black-Scholes price formula for a European call,

$$\Pi_Y(t) = \frac{1}{S(T_0)} c(t, (1-\delta)s, S(T_0), T)$$

$$= \frac{1}{S(T_0)}\left\{(1-\delta)S(t)\Phi(D_1(t)) - S(T_0)e^{-r\tau}\Phi((D_2(t))\right\}$$

where

$$D_1(t) = \frac{1}{\sigma\sqrt{\tau}}(\ln\frac{(1-\delta)S(t)}{S(T_0)} + (r + \frac{\sigma^2}{2})\tau)$$

and

$$D_2(t) = \frac{1}{\sigma\sqrt{\tau}}(\ln\frac{(1-\delta)S(t)}{S(T_0)} + (r - \frac{\sigma^2}{2})\tau).$$

In particular,

$$\Pi_Y(T_0) = (1-\delta)\Phi(D_1(T_0)) - e^{-r(T-T_0)}\Phi((D_2(T_0))$$

where

$$D_1(T_0) = \frac{1}{\sigma\sqrt{T-T_0}}(\ln(1-\delta) + (r + \frac{\sigma^2}{2})(T - T_0))$$

and

$$D_2(T_0) = \frac{1}{\sigma\sqrt{T-T_0}}(\ln(1-\delta) + (r - \frac{\sigma^2}{2})(T-T_0)).$$

Since $\Pi_Y(T_0)$ is non-random (= a numerical constant) we conclude that

$$\Pi_Y(t) = e^{-r(T_0-t)}\left\{(1-\delta)\Phi(D_1(T_0)) - e^{-r(T-T_0)}\Phi((D_2(T_0))\right\} \text{ if } t < T_0.$$

3. (Black-Scholes model) Let $a, K, T > 0$. A financial derivative of European type pays the amount $Y = (\min(S(T)-K, a))^+$ at time of maturity T. Show that the delta of the derivative is positive and does not exceed

$$\frac{\ln(1+\frac{a}{K})}{\sigma\sqrt{2\pi(T-t)}}$$

at time $t < T$.

Solution. Note that $Y = (S(T)-K)^+ - (S(T)-(a+K))^+$. The delta of a call is standard (see Problem 4) and we get that the delta of Y at time t equals

$$\Delta_Y(t) = \Phi(\frac{1}{\sigma\sqrt{\tau}}(\ln\frac{S(t)}{K} + (r+\frac{\sigma^2}{2})\tau) - \Phi(\frac{1}{\sigma\sqrt{\tau}}(\ln\frac{S(t)}{a+K} + (r+\frac{\sigma^2}{2})\tau)$$

where $\tau = T-t$. Hence $\Delta_Y(t) > 0$ since Φ and ln are increasing in the strict sense. Moreover, if

$$\varphi(u) = \frac{1}{\sqrt{2\pi}}\exp(-u^2/2)$$

we have

$$\Delta_Y(t) = \left\{\frac{1}{\sigma\sqrt{\tau}}((\ln\frac{S(t)}{K} + (r+\frac{\sigma^2}{2})\tau) - (\ln\frac{S(t)}{a+K} + (r+\frac{\sigma^2}{2})\tau))\right\}\varphi(\xi).$$

for an appropriate $\xi \in \left]\frac{1}{\sigma\sqrt{\tau}}(\ln\frac{S(t)}{a+K} + (r+\frac{\sigma^2}{2})\tau, \frac{1}{\sigma\sqrt{\tau}}(\ln\frac{S(t)}{K} + (r+\frac{\sigma^2}{2})\tau)\right[$. But $\varphi(\xi) \leq \frac{1}{\sqrt{2\pi}}$ and we get

$$\Delta_Y(t) \leq \frac{1}{\sigma\sqrt{\tau}}(\ln(a+K) - \ln K)\frac{1}{\sqrt{2\pi}}$$

and the result is immediate.

4. (Black-Scholes model) Suppose $t < T$ and $\tau = T - t$. A simple financial derivative of European type with the payoff function $g \in \mathcal{P}$ has the price

$$\Pi_{g(S(T))}(t) = e^{-r\tau} E\left[g(se^{(r-\frac{\sigma^2}{2})\tau+\sigma\sqrt{\tau}G})\right]$$

at time t, where $s = S(t)$ is the stock price at time t and $G \in N(0,1)$.

(a) A European call has the strike price K and determination date T. Show that the call price equals $s\Phi(d_1) - Ke^{-r\tau}\Phi(d_2)$, where

$$d_1 = \frac{1}{\sigma\sqrt{\tau}}(\ln\frac{s}{K} + (r + \frac{\sigma^2}{2})\tau)$$

and $d_2 = d_1 - \sigma\sqrt{\tau}$.

(b) Show that the delta of the call in Part (a) equals $\Phi(d_1)$.

5. (Black-Scholes model) A European call on a US dollar has the strike strike price K and determination date T. Derive the price of the derivative at time t, if the US interest rate equals r_f and the volatility of the exchange rate process, quoted as crowns per dollar, equals σ. As usual the Swedish interest rate is denoted by r.

SOLUTIONS

OPTIONS AND MATHEMATICS

(CTH[mve095], GU[MMA700])

August 29, 2009, morning (4 hours), v
No aids.
Examiner: Christer Borell, telephone number 0705292322
Each problem is worth 3 points.

1. Find a portfolio consisting of European calls and puts with termination date T such that the value of the portfolio at time T equals

$$Y = \min(K, \mid S(T) - K \mid).$$

Solution. By drawing a graph of Y as a function $S(T)$ we get $Y = (K - S(T))^+ + (S(T) - K)^+ - (S(T) - 2K)^+$. Thus a portfolio with long one European put with strike K and expiry T, long one European call with strike K and expiry T, and short one call with strike $2K$ and expiry T will satisfy the requirements in the text.

2. The Black-Scholes call price equals $c = s\Phi(d_1) - Ke^{-r\tau}\Phi(d_2)$, where $\tau = T - t > 0$ and $d_1 = (\sigma\sqrt{\tau})^{-1}\left\{\ln(s/K) + (r + \sigma^2/2)\tau\right\} = d_2 + \sigma\sqrt{\tau}$. Show that

$$\frac{\partial c}{\partial K} = -e^{-r\tau}\Phi(d_2).$$

Solution. Let $\varphi = \Phi'$. We have

$$\frac{\partial c}{\partial K} = s\varphi(d_1)\frac{\partial d_1}{\partial K} - e^{-r\tau}\Phi(d_2) - Ke^{-r\tau}\varphi(d_2)\frac{\partial d_2}{\partial K}$$

$$= -e^{-r\tau}\Phi(d_2) + \frac{\partial d_1}{\partial K}\left\{s\varphi(d_1) - Ke^{-r\tau}\varphi(d_2)\right\}$$

$$= -e^{-r\tau}\Phi(d_2) + \frac{1}{\sqrt{2\pi}}\frac{\partial d_1}{\partial K}\left\{se^{-d_1^2/2} - Ke^{-r\tau}e^{-d_2^2/2}\right\}$$

$$= -e^{-r\tau}\Phi(d_2) + \frac{1}{\sqrt{2\pi}}\frac{\partial d_1}{\partial K}\left\{se^{-d_1^2/2} - Ke^{-r\tau}e^{-d_1^2/2 + d_1\sigma\sqrt{\tau} - \sigma^2\tau/2}\right\}$$

$$= -e^{-r\tau}\Phi(d_2) + \frac{Ke^{-d_1^2/2}}{\sqrt{2\pi}}\frac{\partial d_1}{\partial K}\left\{s/K - e^{d_1\sigma\sqrt{\tau} - (r+\sigma^2/2)\tau}\right\} = -e^{-r\tau}\Phi(d_2).$$

3. Let a be a positive real number and suppose the function $u(t,s)$ satisfies the Black-Scholes differential equation

$$u'_t + \frac{\sigma^2 s^2}{2}u''_{ss} + rsu'_s - ru = 0,\ \ 0 \le t < T,\ \ s > 0.$$

Show that the function $v(t,s) = s^{1-\frac{2r}{\sigma^2}}u(t,\frac{a}{s})$ satisfies the Black-Scholes differential equation.

Solution. We have

$$v'_t(t,s) = s^{1-\frac{2r}{\sigma^2}} u'_t(t, \frac{a}{s})$$

$$v'_s(t,s) = (1 - \frac{2r}{\sigma^2}) s^{-\frac{2r}{\sigma^2}} u(t, \frac{a}{s}) - a s^{-1-\frac{2r}{\sigma^2}} u'_s(t, \frac{a}{s})$$

and

$$v''_{ss}(t,s) = -\frac{2r}{\sigma^2}(1 - \frac{2r}{\sigma^2}) s^{-1-\frac{2r}{\sigma^2}} u(t, \frac{a}{s}) - a(1 - \frac{2r}{\sigma^2}) s^{-2-\frac{2r}{\sigma^2}} u'_s(t, \frac{a}{s})$$

$$+a(1 + \frac{2r}{\sigma^2}) s^{-2-\frac{2r}{\sigma^2}} u'_s(t, \frac{a}{s}) + a^2 s^{-3-\frac{2r}{\sigma^2}} u''_{ss}(t, \frac{a}{s})$$

$$= -\frac{2r}{\sigma^2}(1 - \frac{2r}{\sigma^2}) s^{-1-\frac{2r}{\sigma^2}} u(t, \frac{a}{s}) + a\frac{4r}{\sigma^2} s^{-2-\frac{2r}{\sigma^2}} u'_s(t, \frac{a}{s}) + a^2 s^{-3-\frac{2r}{\sigma^2}} u''_{ss}(t, \frac{a}{s}).$$

Thus

$$v'_t + \frac{\sigma^2 s^2}{2} v''_{ss} + rsv'_s - rv$$

$$= s^{1-\frac{2r}{\sigma^2}}(u'_t(t, \frac{a}{s}) - r(1 - \frac{2r}{\sigma^2}) u(t, \frac{a}{s}) + a2rs^{-1} u'_s(t, \frac{a}{s}) + a^2 \frac{\sigma^2}{2} s^{-2} u''_{ss}(t, \frac{a}{s})$$

$$+r(1 - \frac{2r}{\sigma^2}) u(t, \frac{a}{s}) - ars^{-1} u'_s(t, \frac{a}{s}) - ru(t, \frac{a}{s}))$$

$$= s^{1-\frac{2r}{\sigma^2}}(u'_t(t, \frac{a}{s}) + \frac{\sigma^2}{2}(\frac{a}{s})^2 u''_{ss}(t, \frac{a}{s}) + r\frac{a}{s} u'_s(t, \frac{a}{s}) - ru(t, \frac{a}{s})) = 0.$$

4. Let $(X_n)_{n=1}^{\infty}$ be an i.i.d. such that $P[X_1 = 1] = P[X_1 = -1] = \frac{1}{2}$ and set

$$Y_n = \frac{1}{\sqrt{n}}(X_1 + ... + X_n), \; n \in \mathbf{N}_+.$$

Prove that $Y_n \to G$, where $G \in N(0,1)$.

5. (Dominance principle) Show that the map

$$K \to c(t, S(t), K, T), \; K > 0$$

is convex.

SOLUTIONS
OPTIONS AND MATHEMATICS
(CTH[mve095], GU[MMA700])

January 16, 2010, morning (4 hours), v
No aids.
Examiner: Christer Borell, telephone number 0705292322
Each problem is worth 3 points.

1. (Binomial model, T periods) Set

$$Y = \frac{1}{T}\sum_{t=1}^{T} \ln \frac{S(t)}{S(t-1)}.$$

Prove that $E[Y] = d + p_u(u-d)$ and $\mathrm{Var}(Y) = \frac{1}{T}p_u(1-p_u)(u-d)^2$.

Solution. Using standard notation

$$S(t) = S(t-1)e^{X_t},\ t = 1, ..., T$$

where $X_1, ..., X_T$ are independent and

$$\begin{cases} P[X_t = u] = p_u \\ P[X_t = d] = p_d. \end{cases}$$

Note that

$$E[X_t] = p_u u + p_d d = d + p_u(u-d),$$
$$E\left[X_t^2\right] = p_u u^2 + p_d d^2,$$

and

$$\mathrm{Var}(X_t) = p_u u^2 + p_d d^2 - (p_u u + p_d d)^2$$
$$= p_u(1-p_u)(u^2+d^2) - 2p_u p_d ud = p_u(1-p_u)(u-d)^2.$$

Now since

$$Y = \frac{1}{T}\sum_{t=1}^{T} X_t$$

we have that

$$E[Y] = \frac{1}{T}\sum_{t=1}^{T} E[X_t] = d + p_u(u-d)$$

and

$$\mathrm{Var}(Y) = \frac{1}{T^2}\sum_{t=1}^{T}\mathrm{Var}(X_t) = \frac{1}{T}p_u(1-p_u)(u-d)^2.$$

2. (Black-Scholes model) Let $a, K, T > 0$ be given numbers and consider a simple derivative of European type with time of maturity T and payoff K if $S(T) < a$ and payoff 0 if $S(T) \geq a$. (a) Find the price of the derivative at time $t < T$. (b) Find the delta of the derivative at time $t < T$. (c) Find the vega of the derivative at time $t < T$.

Solution. (a) Set $\tau = T - t$ and let $G \in N(0,1)$. The price of the derivative at time t equals $\pi(t) = v(t, S(t))$, where

$$v(t,s) = e^{-r\tau}E\left[K1_{]0,a[}(se^{(r-\frac{\sigma^2}{2})\tau+\sigma\sqrt{\tau}G})\right]$$

$$= e^{-r\tau}KP\left[se^{(r-\frac{\sigma^2}{2})\tau+\sigma\sqrt{\tau}G} < a\right] = e^{-r\tau}KP\left[G < \frac{\ln\frac{a}{s} - (r-\frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}}\right]$$

$$= e^{-r\tau}K\Phi(\frac{\ln\frac{a}{s} - (r-\frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}}).$$

Hence

$$\pi(t) = e^{-r\tau}K\Phi(\frac{\ln\frac{a}{S(t)} - (r-\frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}}).$$

(b) Let $\varphi = \Phi'$. The delta at time t is given by $\frac{\partial v}{\partial s}_{|s=S(t)}$, where

$$\frac{\partial v}{\partial s} = e^{-r\tau}K\varphi(\frac{\ln\frac{a}{s} - (r-\frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}})\frac{-1}{\sigma\sqrt{\tau}s}.$$

Thus the delta equals

$$-\frac{e^{-r\tau}K}{\sigma\sqrt{\tau}S(t)}\varphi(\frac{\ln\frac{a}{S(t)} - (r-\frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}}).$$

(c) The vega at time t is given by $\frac{\partial v}{\partial \sigma}(t, S(t))$ and equals

$$\begin{aligned} & e^{-r\tau} K \varphi\left(\frac{\ln \frac{a}{S(t)} - (r - \frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}}\right) \left\{ -\frac{\ln \frac{a}{S(t)} - r\tau}{\sigma^2 \sqrt{\tau}} + \frac{\sqrt{\tau}}{2} \right\} \\ = \; & e^{-r\tau} K \left\{ -\frac{\ln \frac{a}{S(t)} - r\tau}{\sigma^2 \sqrt{\tau}} + \frac{\sqrt{\tau}}{2} \right\} \varphi\left(\frac{\ln \frac{a}{S(t)} - (r - \frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}}\right). \end{aligned}$$

3. Suppose $Z = (Z_1(t), Z_2(t))_{t \geq 0}$ is a standard Brownian motion in the plane. Find

$$E\left[\mid Z_1(t) - Z_2(t) \mid e^{(Z_1(t)+Z_2(t))^2} \right] \text{ if } 0 \leq t < \frac{1}{4}.$$

Solution. Note that $(Z_1(t)+Z_2(t), Z_1(t)-Z_2(t))$ is a Gaussian random vector in the plane such that $Z_1(t) \pm Z_2(t) \in N(0, 2t)$. Moreover, since

$$\mathrm{Cov}(Z_1(t) + Z_2(t), Z_1(t) - Z_2(t)) = E\left[(Z_1(t) + Z_2(t))(Z_1(t) - Z_2(t))\right]$$

$$= E\left[(Z_1^2(t) - Z_2^2(t))\right] = t - t = 0.$$

the random variables $Z_1(t)+Z_2(t)$ and $Z_1(t)-Z_2(t)$ are independent. Hence, if $0 \leq t < \frac{1}{4}$ and $G \in N(0,1)$,

$$E\left[\mid Z_1(t) - Z_2(t) \mid e^{(Z_1(t)+Z_2(t))^2} \right]$$

$$= E\left[\mid Z_1(t) - Z_2(t) \mid\right] E\left[e^{(Z_1(t)+Z_2(t))^2} \right]$$

$$= E\left[\mid \sqrt{2t} G \mid\right] E\left[e^{(\sqrt{2t}G)^2} \right] = \sqrt{2t} \int_{\mathbf{R}} \mid x \mid e^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}} \int_{\mathbf{R}} e^{2tx^2} e^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}}$$

$$= 2\sqrt{2t} \int_0^{\infty} x e^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}} \int_{\mathbf{R}} e^{-\frac{(1-4t)x^2}{2}} \frac{dx}{\sqrt{2\pi}}$$

$$= 2\sqrt{\frac{t}{\pi}} \frac{1}{\sqrt{1-4t}} \int_{\mathbf{R}} e^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}} = 2\sqrt{\frac{t}{\pi(1-4t)}}.$$

4. Let $W = (W(t))_{t \geq 0}$ be a standard Brownian motion. (a) Prove that $W(s) - W(t) \in N(0, \mid s - t \mid)$. (b) Suppose a is a strictly positive real number and set $X = (\frac{1}{\sqrt{a}} W(at))_{t \geq 0}$. Prove that X is a standard Brownian motion.

5. (Black-Scholes model) A simple derivative of European type with the payoff $Y = g(S(T))$ at time of maturity T has the price $v(t, S(t))$ at time $t < T$, where

$$v(t,s) = e^{-r\tau} E\left[g(se^{(r - \frac{\sigma^2}{2})\tau + \sigma W(\tau)}) \right]$$

and $\tau = T - t$. Use this formula to find the price of a European styled call with strike price K and time of maturity T.