OPTIONS AND MATHEMATICS (CTH[$mve095$] , GU[$MMA700$])
ASSIGNMENTS 2011
(must be handed in at the latest Friday, April 15, 2011 at 15^{00})

1. (Binomial model with $T = 3$ and $d < r < u$) A financial derivative of European type has the payoff

$$Y = \begin{cases} 1 \text{ if } X_1 = X_2 = X_3 \\ 0 \text{ otherwise} \end{cases}$$

at time of maturity T. (a) Find $\Pi_Y(0)$. (b) The portfolio strategy h replicates Y. Find $h(0) = (h_S(0), h_B(0))$.

2. (Binomial model with T periods and $d < r < u$) Suppose $g(x) = \sum_{k=0}^{n} a_k x^k$, where $a_0, ..., a_n$ are known real numbers and consider a simple financial derivative of European type with the payoff $g(S(T))$ at time of maturity T. Find $\Pi_Y(0)$.

3. Let $0 < a < b < \infty$ and

$$\gamma(x, y) = \begin{cases} -(x-a)(b-y) \text{ if } a \le x \le y \le b, \\ -(b-x)(y-a) \text{ if } a \le y < x \le b, \\ 0 \text{ if } x < a \text{ or } x > b \text{ and } a \le y \le b. \end{cases}$$

(a) Suppose $y \in [a, b]$ is fixed. Find a portfolio $\mathcal{A}_y$ consisting of European puts on S with time of maturity T such that

$$V_{\mathcal{A}_y}(T) = \gamma(S(T), y).$$

(b) The continuous payoff function $g(x)$, $x > 0$, vanishes if $0 < x \le a$ or $x \ge b$ and is twice continuously differentiable in the interval $[a, b]$. Show that

$$g(x) = \int_a^b \gamma(x, y) g''(y) \frac{dy}{b-a}$$

and conclude that

$$g(S(T)) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} g''(y_k) V_{\mathcal{A}_{y_k}}(T)$$

where $y_k = a + k(b-a)/n$, $k = 1, ..., n$.

(Hint for Part (b): First assume $a < x < b$ and show that

$$g(x) = (x-a)g'(x) - \int_a^x (y-a)g''(y)dy$$

and

$$g(x) = (x-b)g'(x) - \int_x^b (b-y)g''(y)dy.$$

The cases when $0 < x \leq a$ or $x \geq b$ are trivial.)

4. Suppose $D = \{(x,y);\ 0 < x < 1 \text{ and } y > 0\}$ and let (X,Y) be a random vector in the plane with the density function $f(x,y) = 1_D(x,y)(4x^3+y)e^{-y}/2$. For which real t is the variance $\mathrm{Var}(X - tY)$ minimal?

5. Suppose (X,Y) is a centred Gaussian random vector in the plane. Show that

$$E\left[X^2Y^2\right] \geq E\left[X^2\right]E\left[Y^2\right].$$

6. Let T be a positive real number and $h = T/n$, where n is a positive integer. Moreover, let $(G_i)_{i=1}^n$ be an i.i.d. with $G_1 \in N(0,1)$ and set

$$X_i = \alpha h + \sigma\sqrt{h}G_i,\ \ i = 1, ..., n,$$

where $(\alpha, \sigma) \in \mathbf{R}\times\,]0,\infty[$ and define

$$\hat{\alpha} = \frac{1}{T}\sum_{i=1}^n X_i \text{ and } \hat{\sigma}^2 = \frac{1}{T}\sum_{i=1}^n X_i^2.$$

Prove that $E\left[\hat{\alpha}\right] = \alpha$, $\mathrm{Var}(\hat{\alpha}) = \sigma^2/T$, $E\left[\hat{\sigma}^2\right] = \sigma^2 + \alpha^2 T/n$, and

$$\mathrm{Var}(\hat{\sigma}^2) = 2\sigma^4/n + 4\alpha^2 T\sigma^2/n^2.$$