SOLUTIONS
OPTIONS AND MATHEMATICS
(CTH[$mve095$], GU[$MMA700$])

May 23, 2011, morning, v.
No aids.
Each problem is worth 3 points.
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1. (Black-Scholes model) Suppose a and b are positive constants. A derivative of European type pays the amount $Y = aS(T) + \frac{b}{S(T)}$ at time of maturity T. (a) Compute the time t price of the derivative. (b) Compute the time t delta of the derivative.

Solution. (a) Set $s = S(t)$. By the weak dominance principle in the Black-Scholes model $\Pi_{S(T)}(t) = s$ and, furthermore, if $\tau = T - t$,

$$\Pi_{\frac{1}{S(T)}}(t) = e^{-r\tau} E\left[\frac{1}{se^{(r-\frac{\sigma^2}{2})\tau+\sigma\sqrt{\tau}G}}\right]$$

where $G \in N(0,1)$. Hence

$$\Pi_{\frac{1}{S(T)}}(t) = e^{-r\tau}(se^{(r-\frac{\sigma^2}{2})\tau})^{-1} E\left[e^{-\sigma\sqrt{\tau}G}\right] =$$

$$\frac{1}{s}e^{(-2r+\frac{\sigma^2}{2})\tau} e^{\frac{\sigma^2\tau}{2}} = \frac{1}{s}e^{(\sigma^2-2r)\tau}$$

and we get

$$\Pi_Y(t) = as + \frac{b}{s}e^{(\sigma^2-2r)\tau}.$$

(b) If $v(t,s) = \Pi_Y(t) = as + \frac{b}{s}e^{(\sigma^2-2r)\tau}$, then

$$\Delta(t) = v'_s(t, S(t)) = a - \frac{b}{S^2(t)}e^{(\sigma^2-2r)\tau}.$$

2. (In this problem give only answers; please, do not hand in any solutions!) Let W be a standard Brownian motion and set $U = W^2(1)$ and $V =$

$W(1)W(2)+W(3)$. Find (a) $E[U]$ (b) $E[V]$ (c) $E[U^2]$ (d) $E[V^2]$ (e) $E[UV]$ (f) $\mathrm{Cov}(U,V)$ and (g) $\mathrm{Cor}(U,V)$.

Solution. Set $X = W(1)$, $Y = W(2) - W(1)$, and $Z = W(3) - W(2)$. Then X, Y, and Z are independent, $X, Y, Z \in N(0,1)$, and

$$U = X^2$$

and

$$V = X^2 + XY + X + Y + Z.$$

(a)
$E[U] = E[X^2] = 1$

(b)
$E[V] = E[X^2] + E[X]E[Y] + E[X] + E[Y] + E[Z] = 1$

(c)
$E[U^2] = E[X^4] = 3$

(d) $E[V^2] = E\left[\{X(X+Y) + (X+Y+Z)\}^2\right] = E[X^2(X^2 + 2XY + Y^2)] + 2E[(X^2 + XY)(X+Y+Z)] + E[(X+Y+Z)^2] = (E[X^4] + E[X^2Y^2]) + 0 + \mathrm{Var}(X+Y+Z) = 3 + 1 + 3 = 7$
Alternative solution:
$E[V^2] = E[X^4] + E[X^2]E[Y^2] + E[X^2] + E[Y^2] + E[Z^2] + 2E[X^3]E[Y] + 2E[X^3] + 2E[X^2]E[Y] + 2E[X^2]E[Z] + E[X^2]E[Y] + E[X]E[Y^2] + E[X]E[Y]E[Z] + E[X]E[Y] + E[X]E[Z] + E[Y]E[Z] = 3 + 1 + 1 + 1 + 1 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 = 7$

(e)
$E[UV] = E[X^4] + E[X^3]E[Y] + E[X^3] + E[X^2]E[Y] + E[X^2]E[Z] = 3 + 0 + 0 + 0 + 0 = 3$

(f)

$\mathrm{Cov}(U,V)=2$

(g)
$\mathrm{Cor}(U,V)=\frac{2}{\sqrt{2}\sqrt{6}}=\frac{1}{\sqrt{3}}$

3. (Black-Scholes model) Suppose $0<a<b$ and $0\le t<T$. A financial derivative of European type pays the amount Y at time of maturity T, where

$$Y=\begin{cases}1 \text{ if } S(T)\in\,]a,b[\,,\\ 0 \text{ if } S(T)\notin\,]a,b[\,.\end{cases}$$

(a) Find $\Pi_Y(t)$. (b) For which value on $S(t)$ is $\Pi_Y(t)$ maximal.

Solution. (a) Let $H(x)=$

$$H_0(x)=\begin{cases}1 \text{ if } x>0,\\ 0 \text{ if } x\le 0\end{cases}\quad \text{and } H_1(x)=\begin{cases}1 \text{ if } x\ge 0,\\ 0 \text{ if } x<0.\end{cases}$$

Then

$$Y=H_0(S(T)-a)-H_1(S(T)-b).$$

Moreover, if $s=S(0)$ and $\tau=T-t$,

$$\Pi_{H_0(S(T)-a)}(t)=e^{-r\tau}\int_{-\infty}^{\infty}H_0(se^{(r-\frac{\sigma^2}{2})\tau-\sigma\sqrt{\tau}x}-a)\varphi(x)dx=$$

$$e^{-r\tau}\int_{-\infty}^{\frac{1}{\sigma\sqrt{\tau}}(\ln\frac{s}{a}+(r-\frac{\sigma^2}{2})\tau)}\varphi(x)dx=e^{-r\tau}\Phi(\frac{1}{\sigma\sqrt{\tau}}(\ln\frac{s}{a}+(r-\frac{\sigma^2}{2})\tau))$$

and, in a similar way,

$$\Pi_{H_1(S(T)-b)}(t)=e^{-r\tau}\Phi(\frac{1}{\sigma\sqrt{\tau}}(\ln\frac{s}{b}+(r-\frac{\sigma^2}{2})\tau)).$$

Thus

$$\Pi_Y(t)=$$

$$e^{-r\tau}\left(\Phi(\frac{1}{\sigma\sqrt{\tau}}(\ln\frac{s}{a}+(r-\frac{\sigma^2}{2})\tau))-\Phi(\frac{1}{\sigma\sqrt{\tau}}(\ln\frac{s}{b}+(r-\frac{\sigma^2}{2})\tau))\right).$$

(b) Set $\Pi_Y(t) = v(s)$. Since $\frac{s}{a} > \frac{s}{b}$ and Φ is strictly increasing, it is obvious that v is a positive function. Moreover, v is continuous and

$$\lim_{s\to\infty} v(s) = \lim_{s\to 0+} v(s) = 0.$$

From this we conclude thar v attains a maximum and the derivative of $v(s)$ vanishes at this point.

We have

$$v'(s) =$$

$$\frac{e^{-r\tau}}{s\sigma\sqrt{\tau}}\left(\varphi(\frac{1}{\sigma\sqrt{\tau}}(\ln\frac{s}{a} + (r - \frac{\sigma^2}{2})\tau)) - \varphi(\frac{1}{\sigma\sqrt{\tau}}(\ln\frac{s}{b} + (r - \frac{\sigma^2}{2})\tau))\right)$$

and, hence $v'(s) = 0$ if and only if

$$(\ln\frac{s}{a} + (r - \frac{\sigma^2}{2})\tau)^2 = (\ln\frac{s}{b} + (r - \frac{\sigma^2}{2})\tau))^2.$$

Thus

$$\ln\frac{s}{a} + (r - \frac{\sigma^2}{2})\tau = \pm(\ln\frac{s}{b} + (r - \frac{\sigma^2}{2})\tau).$$

Here the plus sign leads to $a = b$, which is a contradiction, and we must have

$$2\ln s = \ln ab - 2(r - \frac{\sigma^2}{2})\tau$$

or

$$s = \sqrt{ab}e^{-(r-\frac{\sigma^2}{2})\tau}.$$

4. (Single-period binomial model and $d < r < u$) Let $g : \{S(0)e^u, S(0)e^d\} \to \mathbf{R}$ be a given function and suppose a derivative of European type pays the amount $Y = g(S(1))$ at time 1. Find a portfolio $h = (h_S, h_B)$ which replicates the derivative.

5. Suppose $\alpha \in \mathbf{R}$, $\sigma > 0$ and let

$$S(t) = S(0)e^{\alpha t + \sigma W(t)},\ t \geq 0$$

be a geometric Brownian motion. Moreover, suppose $0 < t_1 < ... < t_n$ and $a_1 < b_1, ..., a_n < b_n$. Prove that

$$P\left[a_1 < S(t_1) < b_1, ..., a_n < S(t_n) < b_n\right]$$

$$= \int_{A_1 \times ... \times A_n} \cdots \int \prod_{k=1}^{n} \left\{ \frac{1}{\sqrt{2\pi(t_k - t_{k-1})}} e^{-\frac{(x_k - x_{k-1})^2}{2(t_k - t_{k-1})}} \right\} dx_1 ... dx_n$$

where $x_0 = 0$, $t_0 = 0$, and

$$A_k = \left] \frac{1}{\sigma}(\ln \frac{a_k}{S(0)} - \alpha t_k), \frac{1}{\sigma}(\ln \frac{b_k}{S(0)} - \alpha t_k) \right[, \ k = 1, ..., n.$$