**SOLUTIONS**
**OPTIONS AND MATHEMATICS**
(CTH[$mve$095], GU[$MMA$700])

January 19, 2013, morning V
No aids.
Each problem is worth 3 points.
Questions on the exam: Christer Borell 0705 292322

1. (Dominance Principle) Consider a forward contract on $S$ with delivery date $T$ and denote by $K$ the forward price $S_{for}^T(0)$ at time zero. A European-style call on $S$ with strike price $K$ and time of maturity $T$ has the time zero price $a$ crowns. Find the time zero price of a European-style put with strike price $K$ and time of maturity $T$.

Solution. By the Put-Call Theorem

$$S(0) - c(0, S(0), K, T) = Ke^{-rT} - p(0, S(0), K, T)$$

and

$$K = S_{for}^T(0) = S(0)e^{rT}.$$

Now as

$$c(0, S(0), K, T) = a$$

we have

$$p(0, S(0), K, T) = a.$$

2. (Black-Scholes model) Consider a European-style derivative paying the amount

$$Y = \sqrt{S(T/2)S(T)}$$

at time of maturity $T$. Find $\Pi_Y(t)$ if $0 \leq t < T$.

Solution.

*Case* $T/2 \le t < T$. Let $\tau = T - t$, $s = S(t)$, and $G \in N(0,1)$. Since $a = S(T/2)$ is known,

$$\Pi_Y(t) = e^{-r\tau} E\left[ (\sqrt{ase^{(r-\frac{\sigma^2}{2})\tau+\sigma\sqrt{\tau}G}} \right]$$

$$= \sqrt{ase^{(r-\frac{\sigma^2}{2})\tau}} e^{-r\tau} E\left[ e^{\frac{\sigma}{2}\sqrt{\tau}G} \right]$$

$$= \sqrt{as} e^{\frac{1}{2}(r-\frac{\sigma^2}{2})\tau - r\tau + \frac{\sigma^2}{8}\tau} = \sqrt{as} e^{-\frac{1}{2}r\tau - \frac{\sigma^2}{8}\tau}$$

$$= b(t)\sqrt{S(t)}$$

where

$$b(t) = \sqrt{S(T/2)} e^{-(\frac{1}{2}r + \frac{\sigma^2}{8})(T-t)}.$$

*Case* $0 \le t < T/2$. We have

$$\Pi_Y(T/2) = S(T/2) e^{-(\frac{1}{4}r + \frac{\sigma^2}{16})T}$$

and, hence,

$$\Pi_Y(t) = S(t) e^{-(\frac{1}{4}r + \frac{\sigma^2}{16})T}.$$

3. Let $Z(t) = (Z_1(t), Z_2(t))$, $t \ge 0$, be a standard Brownian motion in the plane. Find

$$\mathrm{Var}(\max(Z_1(t), Z_2(t))).$$

Solution. Set $X = \max(Z_1(t), Z_2(t))$. Since $Z_1(0) = Z_2(0) = 0$, $\mathrm{Var}(X) = 0$ if $t = 0$. Next assume $t > 0$. The random variables $Z_1(t), Z_2(t) \in N(0,t)$ are independent and

$$P[X \le x] = P[Z_1(t) \le x] P[Z_2(t) \le x] = \Phi^2(x/\sqrt{t}).$$

Hence

$$f_X(x) = 2\varphi(x/\sqrt{t})\Phi(x/\sqrt{t})/\sqrt{t}$$

and

$$E[X] = \int_{-\infty}^{\infty} x f_X(x) dx = 2\sqrt{t} \int_{-\infty}^{\infty} x\varphi(x)\Phi(x) dx$$

$$= 2\sqrt{t}\left\{[-\varphi(x)\Phi(x)]_{-\infty}^{\infty} + \int_{-\infty}^{\infty} \varphi^2(x)dx\right\}$$

$$= 2\sqrt{t}\int_{-\infty}^{\infty} \varphi^2(x)dx$$

$$= 2\sqrt{t}\int_{-\infty}^{\infty} \varphi(\sqrt{2}x)dx/\sqrt{2\pi} = \sqrt{t/\pi}.$$

Moreover,

$$E\left[X^2\right] = \int_{-\infty}^{\infty} x^2 f_X(x)dx = 2t\int_{-\infty}^{\infty} x^2\varphi(x)\Phi(x)dx$$

$$= 2t\left\{[-\varphi(x)x\Phi(x)]_{-\infty}^{\infty} + \int_{-\infty}^{\infty} \varphi(x)(\Phi(x) + x\varphi(x))dx\right\}$$

$$= 2t\int_{-\infty}^{\infty} \varphi(x)(\Phi(x) + x\varphi(x))dx$$

$$= 2t\int_{-\infty}^{\infty} \varphi(x)\Phi(x)dx = t.$$

Thus

$$\mathrm{Var}(X) = t(1 - \frac{1}{\pi}),$$

a formula which also holds if $t = 0$.

4. Consider a single-period binomial model and assume $d < r < u$. A derivative pays the amount $Y = f(X)$ at time 1, where $X = \ln(S(1)/S(0))$. Find a portfolio $h = (h_S, h_B)$ which replicates $Y$.

5. (Black-Scholes model) Suppose $0 \le t < T < \infty$. A simple European-style derivative paying the amount $Y = g(S(T))$ at time of maturity $T$ has the price $v(t, S(t))$ at time $t$, where

$$v(t,s) = e^{-r(T-t)}E\left[g(se^{(r-\frac{\sigma^2}{2})(T-t)+\sigma\sqrt{T-t}G})\right]$$

and $G \in N(0,1)$. Find the time-$t$ price of a European-style call with strike price $K$ and time of maturity $T$.