INTRODUCTION TO THE BLACK-SCHOLES THEORY

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Preface

The aim with this book is to give an introduction to the mathematical theory of financial derivatives based on classical differential and integral calculus. In addition, the presentation requires some knowledge of probability. So called Lebesgue integration and Itô calculus will not be used.

The book is founded on different undergraduate courses given at the Chalmers University of Technology and University of Gothenburg during more than ten years and it is still not quite completed.

Finally I would like to express my deep gratitude to the students in my classes and to Per Hörfelt, Carl Lindberg, and Olaf Torne for suggesting a variety of improvements.

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CHAPTER 1

The Dominance Principle

Introduction

In this book we will study so called financial derivatives, that is financial securities defined in terms of other financial securities. The most common examples of financial derivatives are ordinary stock options.

A call option on a stock is a contract between the writer (or seller) of the call and the buyer of the call. The buyer has the right but not the obligation to buy from the writer the stock at a fixed price called exercise price or strike price. For a European call, the right to buy can only be exercised on the expiration date of the contract. In an American call, the right to buy can be exercised at any time on or before the expiration date of the call. If we replace the right to buy in the definition of a call by the right to sell we get a so called put on the stock. The writer of an option is said to have a short position and the buyer a long position. The expiration date of an option is also called maturity date or termination date.

If an investor borrows and then sells a stock, the investor is said to have a short position in the stock.

Suppose $S(t)$ denotes a stock price at time t and $S = (S(t))_{t\geq 0}$ the corresponding stock price process. A European call on the stock, or for short on S, with the strike price K and the termination date T has the value

$$\max(0, S(T) - K)$$

at the date T and a European put on S with the same strike price and termination date has the value

$$\max(0, K - S(T))$$

at the date T. Here the functions $g_c(x) = \max(0, x - K)$ and $g_p(x) = \max(0, K - x)$ are called payoff functions of the call and the put, respectively.

The payoffs $g_c(S(T))$ and $g_p(S(T))$ are called the intrinsic values of the call and put, respectively.

If g is a real-valued function on the interval $]0, \infty[$, a contract which pays the amount $Y = g(S(T))$ to its owner at maturity T is called a simple European derivative on S with payoff function g. The American version can be exercised at any time t before or on the date T and pays, upon exercise, the amount $g(S(t))$.

A forward contract on S is an agreement to buy or sell the stock at a fixed price K at a given delivery date T. The buyer is said to have a long position and the seller a short position. Initially neither party incurs any costs in entering into the contract. The price K is called delivery price or forward price and is denoted by $S_{for}^T(t)$ if the agreement is made at time t. The payoff to the holder of the long position is $S(T) - K$ and for the short position it is $K - S(T)$.

To simplify the presentation, we will assume a constant interest rate r and suppose there is a bond with the price

$$B(t) = B(0)e^{rt}$$

at time t, where $B(0)$ is a strictly positive real number. The saving account yields the same interest rate and, moreover, it is possible to borrow money at the rate r. If not otherwise stated, it will be assumed that $r > 0$.

Our main concern here is to define prices of financial derivatives from given mathematical assumptions. However, to start with in this chapter we assume that the prices are already defined and, only assuming minimal conditions, we will derive interesting relations between them. In this context, and elsewhere in this presentation, it will be assumed that there are no transaction costs and it is possible to trade in fractions of shares. Furthermore, if not otherwise stated, it is assumed that the stocks do not pay dividends. Option pricing for dividend-paying stocks will be treated in Chapter 7. Options on currencies may be seen as options on dividend-paying assets but we prefer an alternative approach so that these derivatives may be treated very early in the compendium.

1.1 The Dominance Principle

Consider a model with n financial securities $\mathcal{U}_1, ..., \mathcal{U}_n$, where the $n : th$ security is the bond. The price of the $i : th$ security equals $\Pi^{\mathcal{U}_i}(t)$ at time t. A portfolio is an ordered n-tuple of real numbers

$$\mathcal{A} = (a_1, ..., a_n)$$

where a_i is the number of units of asset $\mathcal{U}_i$ (a negative value on a_i means a short position corresponding to $-a_i$ units in $\mathcal{U}_i$). Thus a portfolio is a vector in $\mathbf{R}^n$ and we can define the sum of two portfolios as the corresponding vector sum. In a similar way, we define $\lambda\mathcal{A}$ if $\mathcal{A}$ is a portfolio and $\lambda \in \mathbf{R}$.

The value of a portfolio $\mathcal{A} = (a_1, ..., a_n)$ at time t is, by definition,

$$V_{\mathcal{A}}(t) = \sum_{i=1}^{n} a_i \Pi^{\mathcal{U}_i}(t).$$

Thus

$$V_{\lambda\mathcal{A}+\mu\mathcal{B}}(t) = \lambda V_{\mathcal{A}}(t) + \mu V_{\mathcal{B}}(t)$$

if $\mathcal{A}$ and $\mathcal{B}$ are portfolios and $\lambda, \mu \in \mathbf{R}$.

Below we assume time is restricted to the interval $0 \leq t \leq T$.

Axiom 1. (Dominance Principle) *Suppose $t < T$ is the present time. If the holder of a portfolio $\mathcal{A}$ can ensure that $\mathcal{A}$ exists at time T and $V_{\mathcal{A}}(T) \geq 0$, then $V_{\mathcal{A}}(t) \geq 0$.*

At a first glance, the formulation of Axiom 1 may look complicated but here recall that the writer of an American option cannot prevent the owner from exercising the option before time T. Thus, in general, the holder of a portfolio containing American contracts can not guarantee the existence of the portfolio at a later point of time.

In fact, Axiom 1 is very plausible. To see this assume there is a portfolio $\mathcal{A}$ such that the holder can act so that $V_{\mathcal{A}}(T) \geq 0$. If $V_{\mathcal{A}}(t) < 0$ at the present time t, the investor may add bonds to the portfolio $\mathcal{A}$ in such a proportion that the new portfolio is of no value. Thus by investing the amount zero today the investor, with certainty, has a positive portfolio value at a later date, which is absurd in a mathematical model.

If a European derivative on S pays the amount Y at the termination date T, we denote its price at time t by $\Pi_Y(t)$ or $\Pi_Y(t, T)$ if it is important to emphasize the termination date. The following notation is standard for European call and put prices:

$$\begin{aligned} c(t, S(t), K, T) &= \Pi_{(S(T)-K)^+}(t, T) \\ p(t, S(t), K, T) &= \Pi_{(K-S(T))^+}(t, T) \end{aligned}$$

Here $a^+ = \max(0, a)$ if a is a real number. The price of an American call {put} on S with strike price K and termination date T is denoted by $C(t, S(t), K, T)$ $\{P(t, S(t), K, T)\}$.

Example 1.1.1. Consider a portfolio $\mathcal{A}$ consisting of a long position in an American put on S with strike K and maturity T and, in addition, a short position in its European counterpart. By not exercising the American put, the holder of the portfolio can ensure that the portfolio is exists at time T. Clearly the value of the portfolio then vanishes at time T. Thus, by Axiom 1,

$$P(t, S(t), K, T) - p(t, S(t), K, T) = V_{\mathcal{A}}(t) \geq 0$$

that is,

$$P(t, S(t), K, T) \geq p(t, S(t), K, T).$$

In other words the value of an American put is not smaller than the value of its European counterpart. We leave it as an exercise to prove that

$$p(t, S(t), K, T) \geq 0.$$

Next let us consider a portfolio containing a short position in an American put and a long position in its European counterpart. Now the holder of the portfolio is also the writer of the American put and moreover, if $S(t)$ is small enough (for example $(K - S(t))e^{r(T-t)} > K$) it is not optimal for the owner of the American put to retain this security until the termination date T. Thus Axiom 1 does not apply in this case.

Example 1.1.2. Suppose a portfolio $\mathcal{A}$ consists of a long position in an American call with strike K_0 and termination date T and a short position in an Amercan call on the same underlying asset and termination date but with strike $K_1 \geq K_0$. Suppose the portfolio owner decides to exercise the long as soon as his counterparty exercises the short. As $(S(\lambda)-K_0)^+ \geq (S(\lambda)-K_1)^+$ for all λ it is tempting to conclude that $C(t,S(t),K_0,T)-C(t,S(t),K_1,T) \geq 0$. However, this is not an immediate consequence of Axiom 1 but will follow from Theorem 1.1.3 below.

Theorem 1.1.1. *Suppose $t < T$ is the present time and let $\mathcal{A}$ and $\mathcal{B}$ be portfolios not containing any American options.*

(a) *If $V_{\mathcal{A}}(T) = 0$, then $V_{\mathcal{A}}(t) = 0$.*

(b) *If $V_{\mathcal{A}}(T) = V_{\mathcal{B}}(T)$, then $V_{\mathcal{A}}(t) = V_{\mathcal{B}}(t)$.*

(c) *If $V_{\mathcal{A}}(T) \geq V_{\mathcal{B}}(T)$, then $V_{\mathcal{A}}(t) \geq V_{\mathcal{B}}(t)$.*

PROOF (a) Axiom 1 implies that $V_{\mathcal{A}}(t) \geq 0$. Moreover, $V_{-\mathcal{A}}(T) = -V_{\mathcal{A}}(T) = 0$ and Axiom 1 yields $V_{-\mathcal{A}}(t) \geq 0$ or $-V_{\mathcal{A}}(t) \geq 0$. It follows that $V_{\mathcal{A}}(t) = 0$ and Theorem 1.1(a) is proved.

(b) Part (a) yields $V_{\mathcal{A}-\mathcal{B}}(t) = 0$ and the result follows from the relation $V_{\mathcal{A}-\mathcal{B}}(t) = V_{\mathcal{A}}(t) - V_{\mathcal{B}}(t)$.

(c) The proof follows in a similar way as the proof of Part (b).

Theorem 1.1.2. (Put-Call Parity) *If $t < T$ and $\tau = T - t$, then*

$$S(t) - c(t,S(t),K,T) = Ke^{-r\tau} - p(t,S(t),K,T).$$

PROOF Consider a portfolio $\mathcal{A}$ consisting of a long position in the stock, a long position in the European put on S with strike K and maturity T, a short position in the European call on S with strike K and maturity T, and a short position in the bond corresponding to $K/B(T)$ units. Then

$$V_{\mathcal{A}}(T) = S(T) + (K - S(T))^+ - (S(T) - K)^+ - \frac{K}{B(T)}B(T).$$

Thus if $S(T) \geq K$,

$$V_{\mathcal{A}}(T) = S(T) + 0 - (S(T) - K) - K = 0$$

and if $S(T) < K$,

$$V_{\mathcal{A}}(T) = S(T) + (K - S(T)) - 0 - K = 0$$

and Theorem 1.1.1(a) implies that $V_{\mathcal{A}}(t) = 0$, that is,

$$S(t) + p(t, S(t), K, T) - c(t, S(t), K, T) - Ke^{-r\tau} = 0$$

which is equivalent to the call-put parity relation.

A reader who prefers to avoid short positions in the proof of Theorem 1.1.2 can base the proof on Theorem 1.1.1(b) instead. The details are left to the reader.

Regarding Theorem 1.1.2 and the next theorem recall that the stock does not pay dividends. In this case the next result says that it is never optimal to exercise an American call before expiry. Note however that Chapter 7.2 shows that it may be optimal to exercise an American call option before the maturity date if the underlying asset pays dividends.

Theorem 1.1.3. *If* $t < T$,

$$C(t, S(t), K, T) > S(t) - K$$

and, as a consequence,

$$C(t, S(t), K, T) = c(t, S(t), K, T).$$

Moreover, the map

$$T \to c(t, S(t), K, T)$$

is increasing.

PROOF Since $K > Ke^{-r\tau}$ and $p(t, S(t), K, T) \geq 0$ the Put-Call parity gives

$$c(t, S(t), K, T) > S(t) - K.$$

As in Example 1.1.1 one proves that $C(t, S(t), K, T) \geq c(t, S(t), K, T)$ and hence, $C(t, S(t), K, T) > S(t) - K$. As a consequence, it is not optimal to exercise the American call at time $t < T$.

Suppose the portfolio $\mathcal{A}$ is long one European call option with maturity T_2 and strike price K and short one European call option with maturity T_1 and strike price K, where $T_2 > T_1 \geq t$. The put-call parity gives, as above, that $c(T_1, S(T_1), K, T_2) > S(T_1) - K$ and therefore $c(T_1, S(T_1), K, T_2) \geq c(T_1, S(T_1), K, T_1)$. Thus, $V_{\mathcal{A}}(T_1) \geq 0$ and the dominance principle implies $V_{\mathcal{A}}(t) \geq 0$ for $t \leq T_1$. Hence

$$c(t, S(t), K, T_2) \geq c(t, S(t), K, T_1)$$

and the proof is complete.

A real-valued function f defined on an interval I is said to be convex if

$$f(\theta x + (1-\theta)y) \leq \theta f(x) + (1-\theta) f(y)$$

for all $x, y \in I$ and $0 < \theta < 1$. An affine function $f(x) = ax + b$ is convex and so is the maximum of two convex functions f_1 and f_2 defined on the same interval I. To prove this claim, define $h(z) = \max(f_1(z), f_2(z))$, when $z \in I$. Now if $x, y \in I$ and $0 < \theta < 1$,

$$f_k(\theta x + (1-\theta)y) \leq \theta f_k(x) + (1-\theta) f_k(y), \ \ k = 1, 2$$

and, consequently,

$$f_k(\theta x + (1-\theta)y) \leq \theta h(x) + (1-\theta) h(y), \ \ k = 1, 2$$

and

$$h(\theta x + (1-\theta)y) \leq \theta h(x) + (1-\theta) h(y).$$

In particular, the payoff functions $g_c(s) = \max(0, s-K)$ and $g_p(s) = \max(0, K-s)$ are convex functions of s for fixed K and convex functions of K for fixed s.

Theorem 1.1.4. *The maps*

$$K \to c(t, S(t), K, T), \ \ K > 0$$

and

$$K \to p(t, S(t), K, T), \ \ K > 0$$

are convex.

PROOF Suppose $K_0, K_1 > 0$ and $0 < \theta < 1$. Consider a portfolio $\mathcal{A}$ consisting of a European call on S with strike $\theta K_1 + (1-\theta)K_0$ and maturity T and a portfolio $\mathcal{B}$ consisting of θ units of a European call on S with strike K_1 and maturity T and, in addition, $(1-\theta)$ units of a European call on S with strike K_0 and maturity T. Then, since the function $f(x) = (S(T) - x)^+$ is convex, we get

$$V_{\mathcal{A}}(T) = (S(T) - (\theta K_1 + (1-\theta)K_0))^+$$

$$\leq \theta(S(T) - K_1)^+ + (1-\theta)(S(T) - K_0)^+ = V_{\mathcal{B}}(T).$$

Hence, by Theorem 1.1.1(c), $V_{\mathcal{A}}(t) \leq V_{\mathcal{B}}(t)$, that is,

$$c(t, S(t), \theta K_1 + (1-\theta)K_0, T) \leq$$

$$\theta c(t, S(t), K_1, T) + (1-\theta)c(t, S(t), K_0, T).$$

The convexity in the strike price K of the put price $p(t, S(t), K, T)$ is proved in a similar way or, alternatively, follows from Put-Call Parity and the first part of Theorem 1.1.4.

Finally, in this chapter, we will discuss forward contracts on S. First consider a derivative with payoff function $g(s) = s - K$ and termination date T. If $\mathcal{A}$ is a portfolio with one stock and a short position in the bond of $K/B(T)$ units, $V_{\mathcal{A}}(T) = g(S(T))$. Hence $V_{\mathcal{A}}(t) = \Pi_{g(S(T))}(t)$, that is

$$\Pi_{g(S(T))}(t) = S(t) - Ke^{-r\tau}.$$

If K is chosen such that $\Pi_{g(S(T))}(t) = 0$, we find that

$$S^T_{for}(t) = S(t)e^{r\tau}.$$

A forward contract on a foreign currency, say US dollars, is of a slightly different nature. Let $\xi(t)$ denote the exchange rate at time t. That is, at time t the value of 1 US dollar equals $\xi(t)$ Swedish crowns. First consider a contract with the maturity date T which gives the holder the right and obligation to buy one US dollar at the price of K Swedish crowns. At maturity the value of this contract equals

$$Y = \xi(T) - K$$

Swedish crowns. In the following we assume the US interest rate r_f is strictly positive and constant and let $B_f(t) = B_f(0)e^{r_f t}$ be the price in US dollars of a US bond. Then the process

$$S(t) = B_f(t)\xi(t), \ t \geq 0$$

can be viewed as the price process of a traded Swedish security. Indeed, we may exchange Swedish crowns to US dollars, buy the US bond, and when selling the US bond exchange the cash to Swedish crowns. Now

$$Y = \frac{1}{B_f(T)}(S(T) - B_f(T)K)$$

and we get

$$\Pi_Y(t) = \frac{1}{B_f(T)}(S(t) - B_f(T)Ke^{-r\tau}).$$

If K is chosen such that $\Pi_Y(t) = 0$, K is called the forward price on the US dollar with delivery date T and K is denoted by $\xi_{for}^T(t)$. Thus

$$\xi_{for}^T(t) = \xi(t)e^{(r-r_f)\tau}.$$

To check this price, at time t we borrow the amount $\xi(t)e^{-r_f\tau}$ crowns in a Swedish bank and buy $e^{-r_f\tau}/B_f(T)$ units of the US bond.The value of the bond has grown to 1 US dollar at time T and the amount of debt in the Swedish bank to $\xi(t)e^{(r-r_f)\tau}$ Swedish crowns at time T. This means that, today at time t, we are certain to obtain 1 US dollar at the delivery date T at the price of $\xi(t)e^{(r-r_f)\tau}$ Swedish crowns.

A futures contract is similar to a forward contract but the trading takes place on an exchange, and is subject to regulation.

Exercises

Below it is assumed that the Dominance Principle holds.

1. Suppose $\Delta K > 0$. A butterfly spread on call options pays at the maturity T the amount

$$\max(0, S(T)-K-\Delta K)-2\max(0, S(T)-K)+\max(0, S(T)-K+\Delta K).$$

Show that the value of this option is non-negative at any point of time.

2. Consider a model where $S(t) = B(t)$ for all $t \geq 0$. (a) Prove that $c(t, S(t), K, T) = 0$ if $S(0)e^{rT} < K$. (b) Suppose $K < S(0)e^{rT}$. Prove that $p(t, S(t), K, T) = 0$. (c) Prove that $P(t, S(t), K, T) > 0$ for small $t > 0$ if $S(0) < K$.

3. The price of a contract at time t is N units of currency and it pays at the maturity date $T > t$ the amount

$$N + \alpha N(S(T) - K)^+.$$

Show that

$$\alpha = \frac{1 - e^{-r(T-t)}}{c(t, S(t), K, T)}$$

if $c(t, S(t), K, T) > 0$ and $N \neq 0$.

4. Suppose H is the Heaviside function, i.e. $H(x) = 0$ if $x < 0$ and $H(x) = 1$ if $x \geq 0$, and, moreover, suppose $K, \Delta K$ are positive numbers such that $K - \Delta K > 0$. A digital call option with cash settlement has the payoff function

$$Y_0 = H(S(T) - K)$$

at time of maturity T and a digital call option with physical settlement has the payoff function

$$Y_1 = S(T)H(S(T) - K)$$

at time of maturity T. Digitals option are also known as binary or bet options. Below we assume that the contracts are of European type.

a) Show that

$$c(t, S(t), K, T) - c(t, S(t), K + \Delta K, T) \leq \Delta K \Pi_{Y_0}(t)$$

$$\leq c(t, S(t), K - \Delta K, T) - c(t, S(t), K, T).$$

and conclude that

$$\Pi_{Y_0}(t) = -\frac{\partial c}{\partial K}(t, S(t), K, T),$$

if the derivative exists.

b) Show that

$$\Pi_{Y_1}(t) = c(t, S(t), K, T) + K\Pi_{Y_0}(t).$$

5. Suppose $K_0 \leq K_1$. Show that

$$c(t, S(t), K_1, T) \leq c(t, S(t), K_0, T)$$

and conclude that

$$\frac{\partial}{\partial K} c(t, S(t), K, T) \leq 0$$

if the derivative exists.

6. Prove that

$$c(t, S(t), K, T) \leq S(t)$$

and

$$\lim_{T\to\infty} c(t, S(t), K, T) = S(t).$$

7. Suppose $K_0 \leq K_1$. Prove that

$$c(t, S(t), K_1, T) \geq c(t, S(t), K_0, T) - e^{-r\tau}(K_1 - K_0)$$

and

$$\frac{\partial}{\partial K} c(t, S(t), K, T) \geq -e^{-r\tau}$$

if the derivative exists.

8. Show that

$$\lim_{S(t)\to 0} c(t, S(t), K, T) = 0.$$

9. Show that

$$\frac{\partial c}{\partial s}(t, s, K, T) - \frac{\partial p}{\partial s}(t, s, K, T) = 1$$

if the derivatives exists.

10. Suppose $t_0 < T$ and $n \in \mathbf{N}_+$. Set $h = \frac{1}{n}(T - t_0)$ and $t_i = t_0 + ih$, $i = 1, ..., n$. Moreover, suppose $K > 0$ and consider two European derivatives on S with time of maturity T and payoffs

$$Y_c = \max(0, \frac{1}{n+1} \sum_{i=0}^{n} S(t_i) - K)$$

and

$$Y_p = \max(0, K - \frac{1}{n+1} \sum_{i=0}^{n} S(t_i))$$

respectively. If $t \in [t_{m-1}, t_m[$, show that

$$\frac{e^{-r\tau}}{n+1} \sum_{i=0}^{m-1} S(t_i) + \frac{1 - e^{-r(n-m+1)h}}{1 - e^{-rh}} \frac{S(t)}{n+1} - \Pi_{Y_c}(t)$$

$$= Ke^{-r\tau} - \Pi_{Y_p}(t).$$

Find a similar formula if $t < t_0$. The contracts Y_c and Y_p are usually referred to as Asian call and put options.

11. Below θ_i, $i = 0, \ldots, n$, are positive numbers such that $\Sigma_0^n \theta_i = 1$.

a) Show that if f is convex function on I and $x_i \in I$, $i = 0, \ldots, n$, then

$$f(\Sigma_0^n \theta_i x_i) \leq \Sigma_0^n \theta_i f(x_i).$$

b) Show that a geometric mean is smaller than or equal to an arithmetic mean, i.e. show that if $a_i > 0$, $i = 0, \ldots, n$, then

$$\Pi_0^n a_i^{\theta_i} \leq \Sigma_0^n \theta_i a_i.$$

c) Consider the payoff functions

$$Y_g = \max(0, \Pi_0^n S(t_i)^{\theta_i} - K)$$

and

$$Y_a = \max(0, \Sigma_0^n \theta_i S(t_i) - K),$$

where $t_0 \leq t_1 \leq ... \leq t_n \leq T$ and $K > 0$. Show that if $t \leq t_0$ then

$$\Pi_{Y_g}(t) \leq \Pi_{Y_a}(t) \leq c(t, S(t), K, T).$$

12. Let $t < T$ and $N \in \mathbf{N}_+$. Set $\tau = T - t$, $h = \tau/N$, and $t_n = t + nh$, $n = 0, ..., N$. A financial contract has the following description: at each point of time t_{n-1} the holder of the contract gets a forward contract on S with delivery date t_n and, furthermore at time t_n the holder's saving account adds the amount $S(t_n) - S_{for}^{t_n}(t_{n-1})$ for $n = 1, ..., N$. Prove that the sum of the depositions will grow to the amount $S(T) - S_{for}^{T}(t)$ at time T.

13. Let $0 < a < b < \infty$ and

$$\gamma(x,y) = \begin{cases} -(x-a)(b-y) \text{ if } a \leq x \leq y \leq b, \\ -(b-x)(y-a) \text{ if } a \leq y < x \leq b, \\ 0 \text{ if } x < a \text{ or } x > b \text{ and } a \leq y \leq b. \end{cases}$$

(a) Suppose $y \in [a, b]$ is fixed. Find a portfolio $\mathcal{A}_y$ consisting of European puts on S with time of maturity T such that

$$V_{\mathcal{A}_y}(T) = \gamma(S(T), y).$$

(b) The continuous payoff function $g(x)$, $x > 0$, vanishes if $0 < x \leq a$ or $x \geq b$ and is twice continuously differentiable in the interval $[a, b]$. Show that

$$g(x) = \int_a^b \gamma(x,y) g''(y) \frac{dy}{b-a}$$

and conclude that

$$g(S(T)) = \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} g''(y_k) V_{\mathcal{A}_{y_k}}(T)$$

where $y_k = a + k(b-a)/n$, $k = 1, ..., n$.

(Hint for Part (b): First assume $a < x < b$ and show that

$$g(x) = (x-a)g'(x) - \int_a^x (y-a) g''(y) dy$$

and

$$g(x) = (x-b)g'(x) - \int_x^b (b-y) g''(y) dy.$$

The cases when $0 < x \leq a$ or $x \geq b$ are trivial.)

1.2 Problems with solutions

1. Find a portfolio consisting of European calls and puts with termination date T such that the value of the portfolio at time T equals

$$Y = \min(K, \mid S(T) - K \mid).$$

Solution. By drawing a graph of Y as a function $S(T)$ we get $Y = (K - S(T))^+ + (S(T) - K)^+ - (S(T) - 2K)^+$. Thus a portfolio with long one European put with strike K and expiry T, long one European call with strike K and expiry T, and short one call with strike $2K$ and expiry T will satisfy the requirements in the text.

CHAPTER 2

The Binomial Model

Introduction

In this chapter we will study the so called binomial model introduced in 1979 by Cox, Ross and Rubinstein [CRR] and Rendleman and Bartter [RB].

The binomial model in T periods is a discrete time model where the time index t belongs to the set $\{0, 1, ..., T\}$. The model has two underlying assets, a stock with strictly positive price $S(t)$ at time t and a bond with strictly positive price $B(t)$ at time t. The bond price dynamics is given by

$$B(t+1) = B(t)e^r, \ t = 0, 1, ..., T-1$$

where r is a positive constant called interest rate. To explain the stock price dynamics, we suppose u and d are given reals such that $u > d$ and let $X_1, ..., X_T$ be independent identically distributed random variables such that

$$p_u = P\left[X_t = u\right],$$

$$p_d = P\left[X_t = d\right]$$

and

$$p_u + p_d = 1, \ 0 < p_u < 1.$$

In this presentation, for simplicity, it will be assumed that the event

$$[X_t \notin \{u, d\}]$$

never happens for any t. The stock price dynamics is defined by

$$S(t+1) = S(t)e^{X_{t+1}}, \ t = 0, ..., T-1.$$

Above we have chosen a set up of the binomial model, which will make the step to the famous Black-Scholes model as simple as possible.

2.1. The Single-Period model

In this section we will study the binomial model when $T = 1$ and, for short, X_1 is written X.

Let $h = (h_S, h_B)$ be a portfolio consisting of h_S units of the stock and h_B units of the bond. The portfolio value at time t is given by

$$V_h(t) = h_S S(t) + h_B B(t).$$

The portfolio h is called an arbitrage portfolio if

$$V_h(0) = 0,\ V_h(1) \geq 0,\ \text{and}\ E\left[V_h(1)\right] > 0$$

or, stated otherwise,

$$V_h(0) = 0, V_h(1) \geq 0,\ \text{and}\ V_h(1) \neq 0$$

(here recall, if $f : \Omega \to \mathbf{R}$ is a function the inequality $f \geq 0$ means that $f(\omega) \geq 0$ for all $\omega \in \Omega$ and the relation $f \neq 0$ means that $f(\omega) \neq 0$ for some $\omega \in \Omega$). More explicitly, h is an arbitrage portfolio if

$$h_S S(0) + h_B B(0) = 0$$

and

$$\begin{cases} h_S S(0)e^u + h_B B(0)e^r \geq 0 \\ h_S S(0)e^d + h_B B(0)e^r \geq 0 \end{cases}$$

where strict inequality occurs in at least one of these inequalities.

Theorem 2.1.1. *There exists an arbitrage portfolio in the single-period binomial model if and only if*

$$r \notin \,]d, u[\,.$$

PROOF Suppose $h = (h_S, h_B)$ is an arbitrage portfolio so that

$$h_S S(0) + h_B B(0) = 0$$

and

$$\begin{cases} h_S S(0)e^u + h_B B(0)e^r \geq 0 \\ h_S S(0)e^d + h_B B(0)e^r \geq 0 \end{cases}$$

where strict inequality occurs in at least one of these inequalities. Then

$$h_B B(0) = -h_S S(0)$$

and, hence,

$$\begin{cases} h_S S(0)(e^u - e^r) \geq 0 \\ h_S S(0)(e^d - e^r) \geq 0 \end{cases}$$

where strict inequality occurs in at least one of these inequalities. Then $h_S \neq 0$ and if $h_S > 0$,

$$\begin{cases} e^u - e^r \geq 0 \\ e^d - e^r \geq 0. \end{cases}$$

Thus $r \leq d$ and $r \notin \,]d, u[\,$. The case $h_S < 0$ is treated in a similar way.

Conversely, suppose $r \notin \,]d, u[\,$. Assume first $r \leq d$ and choose $h_S = 1$ and $h_B < 0$ so that

$$S(0) = (-h_B)B(0).$$

Then

$$h_S S(0) + h_B B(0) = 0$$

and

$$\begin{cases} h_S S(0)e^u + h_B B(0)e^r > 0 \\ h_S S(0)e^d + h_B B(0)e^r \geq 0 \end{cases}$$

and it follows that $h = (h_S, h_B)$ is an arbitrage portfolio. The case $r \geq u$ is treated in a similar way.

Next we enlarge the single-period binomial model by adding a European-style derivative security paying the amount $Y = g(S(1))$ to its owner at time of maturity 1. Here the so called payoff function $g : \{S(0)e^u, S(0)e^d\} \to \mathbf{R}$ is a deterministic function. What is a natural price $\Pi_Y(0)$ of this derivative at time 0?

To simplify, set $f(x) = g(S(0)e^x)$, $x \in \{u, d\}$ and note that $Y = f(X)$, where $X = X_1$. In the next step we try to find a portfolio $h = (h_S, h_B)$ consisting of h_S units of the stock and h_B units of the bond which replicates the derivative, that is $V_h(1) = f(X)$ or, what amounts to the same thing,

$$h_S S(0)e^u + h_B B(0)e^r = f(u)$$

and

$$h_S S(0)e^d + h_B B(0)e^r = f(d).$$

From these equations we have

$$h_S S(0) = \frac{f(u) - f(d)}{e^u - e^d}$$

and

$$h_B B(0) = e^{-r}\frac{e^u f(d) - e^d f(u)}{e^u - e^d}.$$

In particular, the quantities h_S and h_B are unique and

$$V_h(0) = h_S S(0) + h_B B(0)$$

$$= e^{-r}\left[q_u f(u) + q_d f(d)\right]$$

where

$$q_u = \frac{e^r - e^d}{e^u - e^d}$$

and

$$q_d = \frac{e^u - e^r}{e^u - e^d}.$$

We define

$$\Pi_Y(0) = V_h(0) = e^{-r}\left[q_u f(u) + q_d f(d)\right].$$

Clearly, $\Pi_{S(1)}(0) = S(0)$ as

$$S(0) = e^{-r}(q_u e^u S(0) + q_d e^d S(0))$$

or

$$q_u e^u + q_d e^d = e^r.$$

Suppose the model is free of arbitrage, which depending on Theorem 2.1.1 means that $u > r > d$. Then $q_u > 0$, $q_d > 0$ and $q_u + q_d = 1$ and the numbers q_u and q_d are called martingale probabilities. Moreover, if we go back to the derivative above with payoff Y and recall that $V_h(1) = Y$ we conclude that the enlarged model is free of arbitrage in the following sense:

if $x, y,$ and z are real numbers satisfying

$$xS(0) + yB(0) + z\Pi_Y(0) = 0$$

then the following cannot occur

$$xS(1) + yB(1) + zY \geq 0 \text{ and } xS(1) + yB(1) + zY \neq 0.$$

Example 2.1.1. Suppose $u > 0 > d$. A derivative has the payoff

$$Y = \max(0, \frac{S(0) + S(1)}{2} - S(0))$$

at time of maturity 1. We want to determine its price at time 0. To this end set $S(0) = s$ so that

$$S(1) = se^X$$

and

$$Y = \max(0, \frac{1}{2}(S(1) - S(0)) = s \max(0, \frac{1}{2}(e^X - 1)) =_{def} f(X).$$

Now

$$f(u) = \frac{s}{2}(e^u - 1)$$

and

$$f(d) = 0.$$

Thus

$$\Pi_Y(0) = e^{-r} q_u \frac{s}{2}(e^u - 1)$$

$$= \frac{se^{-r}}{2}(e^u - 1)\frac{e^r - e^d}{e^u - e^d} = \frac{s}{2}(e^u - 1)\frac{1 - e^{d-r}}{e^u - e^d}.$$

Example 2.1.2. Suppose $d < 0 < r < u$ and consider a call with the payoff $Y = (S(1) - S(0))^+$ at the termination date 1. We want to find the replicating strategy of the derivative at time 0. To solve this problem let $S(0) = s$ and $S(1) = se^X$, where $X = u$ or d. If (h_S, h_B) denotes the replicating strategy at time 0 we have

$$h_S se^u + h_B B(0)e^r = s(e^u - 1)$$

and

$$h_S se^d + h_B B(0)e^r = 0.$$

From this it follows that

$$h_S s(e^u - e^d) = s(e^u - 1)$$

and

$$h_S = \frac{e^u - 1}{e^u - e^d}.$$

Moreover, we get

$$h_B = -\frac{1}{B(0)} h_S s e^{d-r} = \frac{s e^{d-r}}{B(0)} \frac{1 - e^u}{e^u - e^d}.$$

Exercises

1. A derivative has the payoff function

$$g(s) = \max(0, s - K)$$

where

$$S(0)e^d < K < S(0)e^u.$$

Suppose the portfolio $h = (h_S, h_B)$ replicates the derivative. Show that

$$h_S > 0 \text{ and } h_B < 0.$$

2. Suppose

$$S(0)e^d < K < S(0)e^u$$

and consider a European put with the payoff $Y = (K - S(1))^+$ at the termination date 1. Find the replicating strategy of the derivative at time 0. (Answer: $\frac{1}{S(0)} \frac{se^d - K}{e^u - e^d}$ units of the stock and $\frac{e^{u-r}}{B(0)} \frac{K - se^d}{e^u - e^d}$ units of the bond)

3. (Δ-hedging) Assume $u > r > d$ and consider a derivative with payoff $Y = f(X)$ at time 1. Choose a portfolio $\mathcal{D}$ consisting of the derivate and $-\Delta$ units of the stock, where Δ is chosen such that $V_{\mathcal{D}}(1)$ is deterministic.

   (a) Prove that

$$\Delta = \frac{f(u) - f(d)}{S(0)(e^u - e^d)}$$

and, hence,

$$V_{\mathcal{D}}(1) = V_{\mathcal{D}}(0)e^{r}.$$

(b) Give a new motivation of the definition

$$\Pi_Y(0) = e^{-r}(q_u f(u) + q_d f(d)).$$

4. Suppose X is a Rademacher distributed random variable, that is

$$P[X = -1] = P[X = 1] = \frac{1}{2}.$$

Find all $\lambda \in \mathbf{R}$ such that

$$E\left[(a + \lambda bX)^4\right] \le \left(E\left[(a + bX)^2\right]\right)^2$$

for every $a,\, b \in \mathbf{R}$.

5. Suppose $d < r < u$ and consider a call with the payoff $Y = (S(1) - K)^+$ at the termination date 1, where $S(0)e^d < K < S(0)e^u$. (a) Find the call price $\Pi_Y(0)$ at time 0. (b) Prove that $e^{-r}Y > \Pi_Y(0)$ if and only if $S(1) = S(0)e^u$.

2.2. The Multi-Period Model

Set $X = (X_1, ..., X_T)$ and

$$\{u, d\}^T = \{x;\; x = (x_1, ..., x_T) \text{ and } x_t = u \text{ or } d \text{ for } t = 1, ..., T\}.$$

The set $\{u, d\}^T$ has 2^T elements. The range of X may be represented by a $2^T \times T$ matrix here denoted by R_T^{ud}, where the rows correspond to the different realizations of X. Thus

$$R_1^{ud} = \begin{pmatrix} u \\ d \end{pmatrix}$$

$$R_2^{ud} = \begin{pmatrix} u & u \\ u & d \\ d & u \\ d & d \end{pmatrix}$$

$$R_3^{ud} = \begin{pmatrix} u & u & u \\ u & u & d \\ u & d & u \\ u & d & d \\ d & u & u \\ d & u & d \\ d & d & u \\ d & d & d \end{pmatrix}$$

and so on.

Since the random variables $X_1, ..., X_T$ are independent

$$P\left[X_1 = x_1, ..., X_T = x_T\right] = p_{x_1} \cdot ... \cdot p_{x_T}$$

if $x_1, ..., x_T = u$ or d. Therefore if $f : \{u, d\}^T \to \mathbf{R}$,

$$E\left[f(X_1, ..., X_T)\right] = \sum_{x_1,...,x_T = u \text{ or } d} f(x_1, ..., x_T) P\left[X_1 = x_1, ..., X_T = x_T\right]$$

$$= \sum_{x_1,...,x_T = u \text{ or } d} f(x_1, ..., x_T) p_{x_1} \cdot ... \cdot p_{x_T}.$$

Below we will often meet a sum of the type

$$\sum_{x_1,...,x_T = u \text{ or } d} f(x_1, ..., x_T) q_{x_1} \cdot ... \cdot q_{x_T}$$

where q_u and q_d are as in Section 2.1.1 and it is convenient to introduce the notation

$$E^Q\left[f(X_1, ..., X_T)\right] = \sum_{x_1,...,x_T = u \text{ or } d} f(x_1, ..., x_T) q_{x_1} \cdot ... \cdot q_{x_T}.$$

Next recall the stock price dynamics introduced in the Introduction of this chapter. Since

$$S(t) = S(0) e^{X_1 + ... + X_t}$$

the stock price at time t is a deterministic function of $X_1, ..., X_t$ and it is sometimes useful to indicate this by writing

$$S(t) = S(t; X_1, ..., X_t).$$

A sequence $h = (h_S(t), h_B(t))_{t=0}^T$ of pairs of real numbers is called a portfolio strategy (or, for short, strategy) if $h(0) = h(1)$ and $h(t)$ is a deterministic function of $X_1, ..., X_{t-1}$ for every $t \in \{1, ..., T\}$. If so, we sometimes write

$$h_S(t) = h_S(t; X_1, ..., X_{t-1})$$

and

$$h_B(t) = h_B(t; X_1, ..., X_{t-1}).$$

The corresponding value process $V_h = (V_h(t))_{t=0}^T$ is defined by

$$V_h(t) = h_S(t)S(t) + h_B(t)B(t),\ t = 0, 1, .., T.$$

Since $V_h(t)$ is a deterministic function of $X_1, ..., X_t$, we will often write

$$V_h(t) = V_h(t; X_1, ..., X_t)$$

for $t = 0, ..., T$.

We have the picture that a portfolio strategy $h = (h_S(t), h_B(t))_{t=0}^T$ is an investment consisting of $h_S(t)$ units of the stock and $h_B(t)$ units of the bond in the $t : th$ period for $t = 1, ..., T$. Therefore, if

$$V_h(t) = h_S(t+1)S(t) + h_B(t+1)B(t),\ t = 1, ..., T-1$$

the strategy is said to be self-financing. Since $h(0) = h(1)$ the latest relation is also true for $t = 0$, that is,

$$V_h(0) = h_S(1)S(0) + h_B(1)B(0).$$

Theorem 2.2.1. *If $h = (h_S(t), h_B(t))_{t=0}^T$ is a self-financing portfolio strategy,*

$$V_h(0) = e^{-rT} \sum_{x_1,...,x_T = u \text{ or } d} q_{x_1} \cdot ... \cdot q_{x_T} V_h(T; x_1, ..., x_T)$$

that is,

$$V_h(0) = e^{-rT} E^Q \left[V_h(T; X_1, ..., X_T)\right]$$

or

$$V_h(0) = e^{-rT} E^Q \left[V_h(T) \right].$$

PROOF The case $T = 1$ is treated in Section 2.1. Now suppose $T \geq 2$ and Theorem 2.2.1 is true for $T - 1$ periods. Moreover, let $h = (h_S(t), h_B(t))_{t=0}^T$ is a self-financing portfolio strategy. Then, since

$$V_h(1) = h_S(2)S(1) + h_B(2)B(1)$$

we get from the induction assumption that

$$V_h(1; X_1) = e^{-r(T-1)} \sum_{x_2,...,x_T = u \text{ or } d} q_{x_2} \cdot ... \cdot q_{x_T} V_h(T; X_1, x_2, ..., x_T).$$

But

$$V_h(0) = e^{-r} \sum_{x_1 = u \text{ or } d} q_{x_1} V_h(1; x_1)$$

and it follows that

$$\begin{aligned} V_h(0) &= e^{-r} \sum_{x_1 = u \text{ or } d} q_{x_1} \left\{ e^{-r(T-1)} \sum_{x_2,...,x_T = u \text{ or } d} q_{x_2} \cdot ... \cdot q_{x_T} V_h(T; x_1, x_2, ..., x_T) \right\} \\ &= e^{-rT} \sum_{x_1,...,x_T = u \text{ or } d} q_{x_1} \cdot ... \cdot q_{x_T} V_h(T; x_1, ..., x_T). \end{aligned}$$

This concludes the proof of Theorem 2.2.1.

A self-financing strategy $h = (h_S(t), h_B(t))_{t=0}^T$ is said to be an arbitrage strategy if

$$V_h(0) = 0, \; V_h(T) \geq 0, \; \text{and } E\left[V_h(T)\right] > 0$$

or, stated otherwise,

$$V_h(0) = 0, \; V_h(T) \geq 0, \; \text{and } V_h(T) \neq 0.$$

Theorem 2.2.2. *There exists an arbitrage strategy if and only if*

$$r \notin \,]d, u[\, .$$

PROOF First suppose $r \notin \,]d,u[$. By Theorem 2.1.1 there is an arbitrage strategy in the first period. At time 1 the corresponding portfolio is rebalanced and its total value is invested in the bond until time T. Clearly, this gives us an arbitrage portfolio.

Next assume $r \in \,]d,u[$ and consider a self-financing portfolio strategy $h = (h_S(t), h_B(t))_{t=0}^T$ with $V_h(0) = 0$ and $V_h(T) \geq 0$. Then, by applying Theorem 2.2.1,

$$0 = e^{-rT} \sum_{x_1,...,x_T = u \text{ or } d} q_{x_1}...q_{x_T} V_h(T; x_1, ..., x_T)$$

However, if $u > r > d$, then $q_u > 0$ and $q_d > 0$. Hence $V_h(T) = 0$, which completes the proof of Theorem 2.2.2.

From now on it will always be assumed that the model is free of arbitrage, which equivalently means that $u > r > d$.

Suppose $g : \left\{S(0)e^{ku+(T-k)d};\ k = 0, ..., T\right\} \to \mathbf{R}$ is a function. A derivative paying the amount $Y = g(S(T))$ to its owner at maturity T is called a simple European-style derivative and the function g is called payoff function. If the payoff Y at time T is a deterministic function of the stock prices

$$S(0), S(1), ..., S(T)$$

we speak of a European-style contingent claim. For example, a contract ensuring the owner to buy the stock at the lowest price during the time points $\{0, 1, ..., T\}$ is equivalent to a so called lookback option with the payoff

$$Y = S(T) - \min_{t \in \{0,1,...,T\}} S(t)$$

at time T

The payoff of a European-style contingent claim may be written $Y = f(X_1, ..., X_T)$ for an appropriate function $f : \{u, d\}^T \to \mathbf{R}$. What is a natural price $\Pi_Y(t)$ of this contingent claim at time t?

We define $\Pi_Y(T) = Y$. Next suppose $t < T$ and that $\Pi_Y(j)$ has already been defined as a deterministic function of $X_1, ..., X_j$ for $j = T, ..., t+1$. To complete the definition of Π_Y we have to define $\Pi_Y(t)$ as a deterministic function of $X_1, ..., X_t$. To this end we note that there exists a unique portfolio $(h_S(t+1), h_B(t+1))$ in the $(t+1):th$ period, only depending on $X_1, ..., X_t$, which replicates a derivative with payoff $\Pi_Y(t+1)$ at time $t+1$. To be more precise, let

$$\Pi_Y^u(t+1) = \Pi_Y(t+1)_{|X_{t+1}=u}$$

and

$$\Pi_Y^d(t+1) = \Pi_Y(t+1)_{|X_{t+1}=d}.$$

Then

$$h_S(t+1)S(t) = \frac{\Pi_Y^u(t+1) - \Pi_Y^d(t+1)}{e^u - e^d}$$

and

$$h_B(t+1)B(t) = e^{-r}\frac{e^u\Pi_Y^d(t+1) - e^d\Pi_Y^u(t+1)}{e^u - e^d}$$

where, as usual,

$$q_u = \frac{e^r - e^d}{e^u - e^d}$$

and

$$q_d = \frac{e^u - e^r}{e^u - e^d}.$$

Hence

$$\Pi_Y(t) = e^{-r}(q_u\Pi_Y^u(t+1) + q_d\Pi_Y^d(t+1)).$$

Finally, define $h(0) = (h_S(0), h_B(0)) = h(1)$ so that $V_h(t) = \Pi_Y(t)$ for all $t = 0, 1, ..., T$. The above construction gives a self-financing portfolio strategy h with

$$V_h(T) = Y$$

and we say that h replicates the derivative in question. Since every European-style contingent claim is replicable the model is called complete. Note also that

$$\Pi_Y(0) = e^{-rT}E^Q[Y]$$

in view of Theorem 2.2.1.

Exemple 2.2.1. Suppose $u > r > 0$, $d = -u$, $T = 2$, and

$$Y = \max(S(0), S(1), S(2)).$$

We want to determine $\Pi_Y(0)$ and to this end let $S(0) = s$ and remember that

$$S(t+1) = S(t)e^{X_{t+1}},\ t = 0, 1.$$

To simplify, let $v(t) = \Pi_Y(t)$ and we obtain

$$\begin{aligned} v(2)_{|X_1=u,X_2=u} &= \max(s, se^u, se^{u+u}) = se^{2u} \\ v(2)_{|X_1=u,X_2=d} &= \max(s, se^u, se^{u+d}) = se^u \\ v(2)_{|X_1=d,X_2=u} &= \max(s, se^d, se^{d+u}) = s \\ v(2)_{|X_1=d,X_2=d} &= \max(s, se^d, se^{d+d}) = s \end{aligned}$$

and

$$\begin{aligned} v(1)_{|X_1=u} &= e^{-r}(q_u se^{2u} + q_d se^u) \\ v(1)_{|X_1=d} &= e^{-r}(q_u s + q_d s) = e^{-r}s. \end{aligned}$$

Hence

$$\begin{aligned} \Pi_Y(0) &= e^{-r}\left\{q_u e^{-r}(q_u se^{2u} + q_d se^u) + q_d e^{-r}s\right\} \\ &= se^{-2r}\left\{q_u^2 e^{2u} + q_u q_d e^u + q_d\right\} \end{aligned}$$

where, as usual,

$$q_u = \frac{e^r - e^d}{e^u - e^d} = 1 - q_d.$$

Example 2.2.2. Suppose $u > r > d$ and $T = 2$ and consider a European derivative with the payoff Y at time of maturity $T = 2$, where

$$Y = \begin{cases} 0, & \text{if } X_1 = X_2 \\ 1, & \text{otherwise.} \end{cases}$$

We want to find $\Pi_Y(0)$ and $h_S(0)$. To this end set $v(t) = \Pi_Y(t)$ and

$$q_u = \frac{e^r - e^d}{e^u - e^r} = 1 - q_d.$$

Then

$$\begin{cases} v(2)_{|X_1=u,\ X_2=u} = 0 \\ v(2)_{|X_1=u,\ X_2=d} = 1 \\ v(2)_{|X_1=d,\ X_2=u} = 1 \\ v(2)_{|X_1=u,\ X_2=u} = 0 \end{cases}$$

and

$$\begin{cases} v(1)_{|X_1=u} = e^{-r}(q_u \cdot 0 + q_d \cdot 1) = e^{-r}q_d \\ v(1)_{|X_1=d} = e^{-r}(q_u \cdot 1 + q_d \cdot 0) = e^{-r}q_u. \end{cases}$$

Hence
$$\Pi_Y(0) = v(0) = e^{-r}(q_u e^{-r} q_d + q_d e^{-r} q_u) = 2e^{-2r} q_u q_d.$$

Futhermore, as $h(0) = h(1)$,

$$\begin{cases} h_S(0)S(0)e^u + h_B(0)B(0)e^r = v(1)_{|X_1=u} \\ h_S(0)S(0)e^d + h_B(0)B(0)e^r = v(1)_{|X_1=d} \end{cases}$$

or
$$\begin{cases} h_S(0)S(0)e^u + h_B(0)B(0)e^r = e^{-r}q_d \\ h_S(0)S(0)e^d + h_B(0)B(0)e^r = e^{-r}q_u \end{cases}$$

and it follows that
$$h_S(0) = e^{-r}\frac{q_d - q_u}{S(0)(e^u - e^d)}.$$

Again consider a European-style contingent claim with the payoff $Y = f(X_1, ..., X_T)$ at time T and choose a replicating portfolio strategy h so that $V_h(T) = Y$. Then, by Theorem 2.2.2,

$$\Pi_Y(0) = e^{-rT} \sum_{x_1,...,x_T = u \text{ or } d} q_{x_1} \cdot ... \cdot q_{x_T} f(x_1, ..., x_T).$$

To simplify this formula set

$$R_T^{10} = (a_{jk})_{1\le j\le 2^T, 1\le k\le T}\ .$$

Moreover, let r_j be the $j:th$ row of the matrix R_T^{10}, let

$$b_j = (u-d)r_j + [d \ldots d]$$

and let n_j number of $u:s$ in the vector b_j, that is

$$n_j = \sum_{k=1}^{T} a_{jk}$$

for $j = 1, ..., 2^T$. Then

$$\Pi_Y(0) = e^{-rT} \sum_{j=1}^{2^T} q_u^{n_j} q_d^{T-n_j} f(b_j).$$

Here the sum in this series contains 2^T terms, which is an extremely large number even if T is of a rather moderate size.

A simple derivative with payoff $Y = g(S(T)) = g(S(0)\exp(X_1+...+X_T))$ is much simpler to handle numerically. Since there are

$$\left(\begin{array}{c} T \\ k \end{array} \right) = \frac{T!}{k!(T-k)!}$$

sequences of length T that have exactly k $u:s$, we have

$$\Pi_Y(0) = e^{-rT} \sum_{k=0}^{T} \left(\begin{array}{c} T \\ k \end{array} \right) q_u^k q_d^{T-k} g(S(0)e^{ku+(T-k)d}).$$

Setting $\tau = T - t$, note that

$$\Pi_Y(t) = e^{-r\tau} \sum_{k=0}^{\tau} \left(\begin{array}{c} \tau \\ k \end{array} \right) q_u^k q_d^{\tau-k} g(S(t)e^{ku+(\tau-k)d})$$

which implies that $\Pi_Y(t)$ is a deterministic function of $S(t)$. In connection, with computations it is often preferable to proceed recursively. Writing $\Pi_Y(t) = v(t, S(t))$,

$$v(T, S(0)e^{ku+(T-k)d}) = g(S(0)e^{ku+(T-k)d}),\ k = 0, ..., T$$

and for every $t = T - 1, ..., 1, 0,$

$$v(t, S(0)e^{ku+(t-k)d})$$

$$= e^{-r}(q_u v(t+1, S(0)e^{(k+1)u+(t-k)d}) + q_d v(t+1, S(0)e^{ku+(t+1-k)d}))$$

for $k = 0, ..., t$. We get $\Pi_Y(0) = v(0, S(0))$.

Next let $k \in \{0, ..., T\}$ be fixed and define

$$AD_k = \begin{cases} 1 \text{ if } S(T) = S(0)e^{ku+(T-k)d} \\ 0 \text{ if } S(T) \neq S(0)e^{ku+(T-k)d} \end{cases}$$

so that

$$\Pi_{AD_k}(0) = e^{-rT} \left(\begin{array}{c} T \\ k \end{array} \right) q_u^k q_d^{T-k}.$$

The quantities $\Pi_{AD_k}(0)$, $k = 0, 1, ..., T$, are called the Arrow-Debreu prices. For a simple European-style derivative with payoff $Y = g(S(T))$ at time of maturity T,

$$\Pi_Y(0) = \sum_{k=0}^{T} \Pi_{AD_k}(0) g(S(0)e^{ku+(T-k)d})$$

(the Arrow-Debreu prices serve the same role a Green function in mathematical physics).

Finally in this section we consider so called American contingent claims.

For any $t = 0, 1, ..., T$, suppose Y_t is a deterministic function of $X_1, ..., X_t$ given by the equation

$$Y_t = f_t(X_1, ..., X_t)$$

where

$$f_t : \{u, d\}^t \to \mathbf{R}.$$

Here Y_0 is a real number known at time 0. A contingent claim of American type with payoff process $Y = (Y_t)_{t=0}^{T}$ gives its owner the right to exercise the contract at any time point $t \in \{0, 1, ..., T\}$ and, if so, the contract pays the amount Y_t to its owner and expires at the same time. What is a natural price $\Pi_Y(t)$ of this derivative at time t if the asset exists at this moment? As above we will make use of the notation

$$\Pi_Y^u(t+1) = \Pi_Y(t+1)_{|X_{t+1}=u}$$

and

$$\Pi_Y^d(t+1) = \Pi_Y(t+1)_{|X_{t+1}=d}.$$

First we define $\Pi_Y(T) = Y_T$. If

$$e^{-r}(q_u \Pi_Y^u(T) + q_d \Pi_Y^d(T)) > Y_{T-1}$$

it is not optimal for the owner to exercise the derivative at time $T - 1$. We therefore define

$$\Pi_Y(T-1) = \max(Y_{T-1}, e^{-r}(q_u \Pi_Y^u(T) + q_d \Pi_Y^d(T))$$

and, in general,

$$\Pi_Y(t) = \max(Y_t, e^{-r}(q_u \Pi_Y^u(t+1) + q_d \Pi_Y^d(t+1))), \ t = T, T-1, ..., 0.$$

Again, in many interesting cases it is impossible to compute $\Pi_Y(0)$ since 2^T is an extemely larege number even if T is of a rather moderate size. The situation is much simpler if each Y_t is a deterministic function of $S(t)$. Introducing

$$Y_t = g_t(S(t))$$

we find that $\Pi_Y(t)$ is a deterministic function of function $v(t, S(t))$ of $(t, S(t))$ and

$$v(T, S(0)e^{ku+(T-k)d}) = g_T(S(0)e^{ku+(T-k)d}),\ k = 0, ..., T.$$

Moreover, for every $t = T-1, ..., 1, 0,$

$$v(t, S(0)e^{ku+(t-k)d})$$

equals

$$\max(g_t(S(0)e^{ku+(t-k)d}), e^{-r}(q_u v(t+1, S(0)e^{(k+1)u+(t-k)d}) + q_d v(t+1, S(0)e^{ku+(t+1-k)d}))$$

for each $k = 0, ..., t$.

Exercises

1. A European derivative pays the amount Y at time of maturity $T = 2$, where
$$Y = \begin{cases} 0, & \text{if } X_1 = X_2 \\ 1, & \text{otherwise.} \end{cases}$$
(a) Find the price $\Pi_Y(0)$ of the derivative at time zero. (b) Suppose $(h_S(t), h_B(t))_{t=0}^T$ is a self-financing portfolio which replicates the derivative. Find $h_S(0)$.

2. Suppose $T = 2$ and consider a European-style contingent claim with the payoff
$$Y = \max(0, (S(0)S(1)S(2))^{\frac{1}{3}} - K)$$
at time of maturity 2, where K is a real number satisfying
$$S(0)e^d < K \leq S(0)e^{\frac{1}{3}u+\frac{2}{3}d}.$$
Show that
$$\Pi_Y(0) = e^{-2r}\left[S(0)q_u^2 e^u + S(0)q_u q_d(e^{\frac{2}{3}u+\frac{1}{3}d} + e^{\frac{1}{3}u+\frac{2}{3}d}) - q_u(1+q_d)K\right].$$

3. Suppose $T = 2$ and consider a European-style contingent claim with the payoff
$$\max\left(0, \frac{1}{3}(S(0) + S(1) + S(2)) - K\right)$$
at time of maturity 2, where K is a real number satisfying
$$\frac{S(0)}{3}(1 + e^d + e^{2d}) < K < \frac{S(0)}{3}(1 + e^d + e^{u+d}).$$
Find $\Pi_Y(0)$.

4. A European-style contingent claim pays
$$Y = S(T) - \min_{t\in\{0,1,\ldots,T\}} S(t)$$
at time T. Find $\Pi_Y(0)$ if $u = -d = 0.1$, $r = 0.05$ and

(a) $T = 1$ (Answer : 0.0731 $S(0)$)

(b) $T = 2$ (Answer: 0.124 $S(0)$)

5. Suppose $u > r > 0 \geq d$. A European-style contingent claim with termination date T has payoff $Y = S(T)$ if $S(0) < S(1) < \ldots < S(T)$ and $Y = S(0)$ otherwise. Find $\Pi_Y(0)$.

(Answer: $e^{-rT}\left\{1 + \left(\frac{e^r - e^d}{e^u - e^d}\right)^T (e^{Tu} - 1)\right\} S(0)$)

6. Let $h = (h_S(t), h_B(t))_{t=0}^T$ be a portfolio strategy. The gain process $(G(t))_{t=0}^T$ is defined by $G(0) = 0$ and
$$\begin{aligned} G(t) &= h_S(1)(S(1) - S(0)) + \ldots + h_S(t)(S(t) - S(t-1) \\ &\quad + h_B(1)(B(1) - B(0)) + \ldots + h_B(t)(B(t) - B(t-1)) \end{aligned}$$
for $t = 1, \ldots, T$. Prove that h is self-financing if and only if
$$V_h(t) = V_h(0) + G(t),\ t = 0, \ldots, T.$$

2.3 Problems with solutions

1. (The binomial model with $u > 0$, $d = -u$, $r = \frac{1}{2}u$, and $T = 2$). Suppose $g(x) = 1$ if $x = 0$ and $g(x) = 0$ if $x \neq 0$. A European-style derivative has the payoff $g(S(T) - S(0))$ at time of maturity T. (a) Find the price of the derivative at time 0. (b) Suppose the strategy h replicates the derivative. Find $h_S(0)$. The answers in Parts (a) and (b) may contain the martingale probabilities q_u and q_d.

Solution. (a) We have

$$q_u = \frac{e^r - e^d}{e^u - e^d} = \frac{e^{u/2} - e^{-u}}{e^u - e^{-u}}$$

and

$$q_d = 1 - q_u = \frac{e^u - e^{u/2}}{e^u - e^{-u}}.$$

Thus if $v(t)$ denotes the price of the derivative at time t,

$$\begin{aligned} v(2)_{|X_1=u,X_2=u} &= 0 \\ v(2)_{|X_1=u,X_2=d} &= 1 \\ v(2)_{|X_1=d,X_2=u} &= 1 \\ v(2)_{|X_1=d,X_2=d} &= 0 \end{aligned}$$

and

$$\begin{aligned} v(1)_{|X_1=u} &= e^{-r}(q_u 0 + q_d 1) = e^{-r} q_d \\ v(1)_{|X_1=d} &= e^{-r}(q_u 1 + q_d 0) = e^{-r} q_u. \end{aligned}$$

Now

$$\begin{aligned} v(0) &= e^{-r}(q_u e^{-r} q_d + q_d e^{-r} q_u) \\ &= 2e^{-2r} q_u q_d = 2e^{-u} q_u q_d. \end{aligned}$$

(b) Recall that $h(0) = h(1)$ and

$$\begin{aligned} h_S(1)S(0)e^u + h_B(1)B(0)e^r &= e^{-r} q_d \\ h_S(1)S(0)e^d + h_B(1)B(0)e^r &= e^{-r} q_{u.} \end{aligned}$$

Hence

$$h_S(0) = h_S(1) = e^{-u/2} \frac{1}{S(0)} \frac{q_d - q_u}{e^u - e^{-u}}.$$

2. (The single-period binomial model with $p_u = \frac{1}{2}$) Suppose $X = \ln \frac{S(1)}{S(0)}$. Show that

$$u = E[X] + \sqrt{\operatorname{Var}(X)}$$

and

$$d = E[X] - \sqrt{\operatorname{Var}(X)}\,.$$

Solution. We have

$$E[X] = \frac{1}{2}u + \frac{1}{2}d$$

and

$$E\left[X^2\right] = \frac{1}{2}u^2 + \frac{1}{2}d^2.$$

Consequently,

$$\operatorname{Var}(X) = \frac{1}{4}(u-d)^2$$

and it follows that

$$E[X] + \sqrt{\operatorname{Var}(X)} = \frac{1}{2}u + \frac{1}{2}d + \frac{1}{2}(u-d) = u$$

and

$$E[X] - \sqrt{\operatorname{Var}(X)} = \frac{1}{2}u + \frac{1}{2}d - \frac{1}{2}(u-d) = d.$$

3. (Binomial model) Suppose $d = -u$ and $e^r = \frac{1}{2}(e^u + e^d)$. A European-style financial derivative has the maturity date $T = 4$ and payoff $Y = f(X_1 + X_2 + X_3 + X_4)$, where $f(x) = 1$ if $x \in \{4u, 0, -4u\}$ and $f(x) = -1$ if $x \in \{2u, -2u\}$. Show that $\Pi_Y(0) = 0$.

Solution. It follows that $d < r < u$ and

$$q_u = \frac{e^r - e^d}{e^u - e^d} = \frac{e^u - e^r}{e^u - e^d} = q_d.$$

Hence $q_u = q_d = \frac{1}{2}$. Furthemore,

$$\Pi_Y(0) = e^{-4r} \sum_{k=0}^{4} \left(\begin{array}{c} 4 \\ k \end{array} \right) q_u^k q_d^{4-k} f(ku + (4-k)d)$$

$$= e^{-4r} \sum_{k=0}^{4} \left(\begin{array}{c} 4 \\ k \end{array} \right) q_u^k q_d^{4-k} f((2k-4)u)$$

$$= e^{-4r} (\frac{1}{2})^4 (1 - 4 + 6 - 4 + 1) = 0.$$

4. (Binomial model) Suppose $T = 3$, $u > r > 0$, and $d = -u$. A European-style derivative has the payoff Y at time of maturity T, where

$$Y = \begin{cases} 1, & \text{if } X_1 = X_2 = X_3, \\ 0, & \text{otherwise.} \end{cases}$$

Find $\Pi_Y(0)$ (the answer may contain the martingale probabilities q_u and q_d, which must, however, be defined explicitely).

Solution. We have

$$q_u = \frac{e^r - e^{-u}}{e^u - e^{-u}} \text{ and } q_d = \frac{e^u - e^r}{e^u - e^{-u}}.$$

Introducing $\Pi_Y(t) = v(t)$, it follows that

$$\begin{cases} v(2)_{|X_1=u,X_2=u} = e^{-r}(q_u \cdot 1 + q_d \cdot 0) = e^{-r} q_u \\ v(2)_{|X_1=u,X_2=d} = e^{-r}(q_u \cdot 0 + q_d \cdot 0) = 0 \\ v(2)_{|X_1=d,X_2=u} = e^{-r}(q_u \cdot 0 + q_d \cdot 0) = 0 \\ v(2)_{|X_1=d,X_2=d} = e^{-r}(q_u \cdot 0 + q_d \cdot 1) = e^{-r} q_d \end{cases}$$

and

$$\begin{cases} v(1)_{|X_1=u} = e^{-r}(q_u e^{-r} q_u + q_d \cdot 0) = e^{-2r} q_u^2 \\ v(1)_{|X_1=d} = e^{-r}(q_u \cdot 0 + q_d e^{-r} q_d) = e^{-2r} q_d^2. \end{cases}$$

Thus

$$v(0) = e^{-r}(q_u e^{-2r} q_u^2 + q_d e^{-2r} q_d^2) = e^{-3r}(q_u^3 + q_d^3).$$

Alternative solution. We have $Y = 1_{\{S(0)e^{3u}, S(0)e^{-3u}\}}(S(3))$ and the derivative is simple. Hence

$$\Pi_Y(0) = e^{-3r} \sum_{k=0}^{3} \left(\begin{array}{c} 3 \\ k \end{array} \right) q_u^k q_d^{3-k} 1_{\{S(0)e^{3u}, S(0)e^{-3u}\}}(S(0)e^{ku+(3-k)(-u)})$$

$$= e^{-3r} \sum_{k \in \{0,3\}} \left(\begin{array}{c} 3 \\ k \end{array} \right) q_u^k q_d^{3-k} = e^{-3r}(q_u^3 + q_d^3).$$

5. (The one period binomial model, where $d < 0 < r < u$) Consider a put with the payoff $Y = (S(0) - S(1))^+$ at the termination date 1. Find the replicating strategy of the derivative at time 0.

Solution: Let $S(0) = s$ and $S(1) = se^X$, where $X = u$ or d. If (h_S, h_B) denotes the replicating strategy at time 0 we have

$$h_S s e^u + h_B B(0) e^r = 0$$

and

$$h_S s e^d + h_B B(0) e^r = s(1 - e^d).$$

From this it follows that

$$h_S s(e^u - e^d) = s(e^d - 1)$$

and

$$h_S = \frac{e^d - 1}{e^u - e^d}.$$

Moreover, we get

$$h_B = -\frac{1}{B(0)} h_S s e^{u-r} = \frac{se^{u-r}}{B(0)} \frac{1 - e^d}{e^u - e^d}.$$

6. (Binomial model; T periods, $d < 0 < r < u$) A European-style financial derivative pays the amount Y at time of maturity T, where

$$Y = \begin{cases} 0 \text{ if } S(T-1) \leq S(T) \\ 1 \text{ if } S(T-1) > S(T). \end{cases}$$

Find a self-financing portfolio strategy $h = (h_S(t), h_B(t))_{t=0}^T$ which replicates Y.

Solution. Set $\Pi_Y(t) = v(t)$,

$$q_u = \frac{e^r - e^d}{e^u - e^d} \text{ and } q_d = \frac{e^u - e^r}{e^u - e^d}.$$

We have $v(T-1, X_1 = x_1, ..., X_{T-1} = x_{T-1}) = e^{-r}(q_u \cdot 0 + q_d \cdot 1) = e^{-r} q_d$ for all $x_1, ..., x_{T-1} \in \{u, d\}$. Hence

$$v(t) = e^{-(T-1-t)r} e^{-r} q_d = e^{-(T-t)r} q_d \text{ if } 0 \le t \le T-1.$$

Now

$$h_S(t; x_1, ..., x_{t-1}) = 0 \text{ and } h_B(t; x_1, ..., x_{t-1}) = \frac{e^{-Tr} q_d}{B(0)} \text{ if } 1 \le t \le T-1$$

and, as usual, $h(0) = h(1)$. Moreover,

$$\begin{cases} h_S(T; x_1, ..., x_{T-1}) S(T-1) e^u + h_B(T; x_1, ..., x_{T-1}) B(T-1) e^r = 0 \\ h_S(T; x_1, ..., x_{T-1}) S(T-1) e^d + h_B(T; x_1, ..., x_{T-1}) B(T-1) e^r = 1 \end{cases}$$

and we get

$$\begin{cases} h_S(T; x_1, ..., x_{T-1}) = -\frac{1}{S(T-1)(e^u - e^d)} \\ h_B(T; x_1, ..., x_{T-1}) = \frac{e^{u-r}}{B(T-1)(e^u - e^d)}. \end{cases}$$

7. (Binomial model with T periods and $u > r > d$) A European-style financial derivative pays the amount $Y = \sqrt{S(T)}$ at time of maturity T. Find $\Pi_Y(0)$.

Solution. Set

$$q_u = \frac{e^r - e^d}{e^u - e^d} \text{ and } q_d = \frac{e^u - e^r}{e^u - e^d}.$$

We have

$$\Pi_Y(0) = e^{-rT} \sum_{k=0}^{T} \binom{T}{k} q_u^k q_d^{T-k} \sqrt{S(0) e^{ku + (T-k)d}} =$$

$$e^{-rT}\sqrt{S(0)}\sum_{k=0}^{T}\left(\begin{array}{c} T \\ k \end{array}\right)(q_u e^{u/2})^k (q_d e^{d/2})^{(T-d)} =$$

$$e^{-rT}\sqrt{S(0)}(q_u e^{u/2} + q_d e^{d/2})^T.$$

8. (Binomial model in T period with $d < r < u$) A European-style financial derivative pays the amount

$$Y = \ln \frac{S(T)}{S(0)}$$

at time of maturity T. Find $\Pi_Y(0)$.

Solution. Using standard notation,

$$Y = \ln \frac{S(T-1)}{S(0)} + X_T$$

and, hence,

$$\Pi_Y(T-1) = e^{-r} \ln \frac{S(T-1)}{S(0)} + e^{-r}(q_u u + q_d d)$$

where

$$q_u = \frac{e^r - e^d}{e^u - e^r} = 1 - q_d.$$

In a similar way, if $T \geq 2$,

$$\Pi_Y(T-2) = e^{-r}(e^{-r} \ln \frac{S(T-2)}{S(0)} + e^{-r}(q_u u + q_d d)) + e^{-2r}(q_u u + q_d d)$$

$$= e^{-2r} \ln \frac{S(T-2)}{S(0)} + 2e^{-2r}(q_u u + q_d d)$$

and by iteration

$$\Pi_Y(0) = Te^{-Tr}(q_u u + q_d d).$$

9. Suppose $T = 2$, $e^r = \frac{1}{2}(e^u + e^d)$, and $B(2) = 1$. A European-style derivative pays the amount

$$Y = \mid \frac{S(2)}{S(1)} - \frac{S(1)}{S(0)} \mid$$

at time of maturity T and the self-financing portfolio strategy $h(t) = (h_S(t), h_B(t))$, $t = 0, 1, 2$, replicates Y. Find q_u, $h_S(1)$, and $h_B(2; d)$.

Solution. *Computation of* q_u. We have that

$$q_u = \frac{e^r - e^d}{e^u - e^d} = \frac{\frac{1}{2}(e^u + e^d) - e^d}{e^u - e^d} = \frac{1}{2}.$$

Computation of $h_S(1)$. Set $\Pi_Y(t) = v(t)$. Then

$$\begin{cases} v(2)_{|X_1=u,\ X_2=u} = 0 \\ v(2)_{|X_1=u,\ X_2=d} = e^u - e^d \\ v(2)_{|X_1=d,\ X_2=u} = e^u - e^d \\ v(2)_{|X_1=d,\ X_2=d} = 0 \end{cases}$$

and it follows that $v(1) = \frac{1}{2}e^{-r}(e^u - e^d)$. Thus $h_S(1) = 0$.

Computation of $h_B(2; d)$. For short, set $\kappa_S = h_S(2; d)$ and $\kappa_B = h_B(2; d)$. Now

$$\begin{cases} \kappa_S S(0)e^d e^u + \kappa_B B(0)e^r e^r = e^u - e^d \\ \kappa_S S(0)e^d e^d + \kappa_B B(0)e^r e^r = 0 \end{cases}$$

and

$$\kappa_B = h_B(2; d) = -\frac{e^d}{B(0)e^{2r}} = -e^d.$$

CHAPTER 3

Review of Basic Concepts in Probability

Introduction

If you intend to buy a share of a stock at time 0 at the price $S(0)$ and have the time horizon T it is interesting to know the return

$$R = \frac{S(T) - S(0)}{S(0)}$$

of the investment during the time interval $[0, T]$. However, R is not known until time T and, before the investment, it is natural to view the return as a random variable. In fact, probability seems to be an inevitable tool in finance.

The purpose of this chapter is to recall some basic definitions in probability theory and to go a little bit further than in the previous chapter. The approach is rather intuitive and will not be based on measure theory. However, certain results stated below require measure theory for their proofs.

3.1 Basic Concepts

Consider a fixed sample space Ω. An event is a subset of Ω and a random variable is a map from Ω into the real numbers. The probability of an event A is denoted by $P[A]$ and the expectation of a random variable X by $E[X]$. If A is an event we define a random variable 1_A by setting

$$1_A = \begin{cases} 1 \text{ if } A \text{ occurs} \\ 0 \text{ if } A \text{ does not occur.} \end{cases}$$

Thus

$$P[1_A = 1] = P[A]$$

and

$$P[1_A = 0] = 1 - P[A].$$

Hence

$$E[1_A] = 0 \cdot (1 - P[A]) + 1 \cdot P[A] = P[A].$$

If X is a random variable with a finite second order moment $E[X^2] < \infty$ it is interesting to note that $E[\mid X \mid] < \infty$. Indeed,

$$\begin{aligned} E[\mid X \mid] &= E\left[\mid X \mid 1_{[\mid X \mid \leq 1]}\right] + E\left[\mid X \mid 1_{[\mid X \mid > 1]}\right] \\ &\leq E[1] + E\left[X^2\right] = 1 + E\left[X^2\right] < \infty. \end{aligned}$$

Moreover, in this case, we define the variance of X by

$$\operatorname{Var}(X) = E\left[(X - E[X])^2\right]$$

and it follows that

$$\operatorname{Var}(X) = E\left[X^2\right] - (E[X])^2.$$

The variance of X is a measure on how much X deviates from its expectation. If c is a real number X and $X + c$ have the same variance.

If $a > 0$, the important Markov inequality states that

$$P[\mid X \mid \geq a] \leq \frac{1}{a} E[\mid X \mid].$$

The proof is simple. First

$$1_{[\mid X \mid \geq a]} \leq \frac{1}{a} \mid X \mid$$

and by taking the expectation

$$E\left[1_{[\mid X \mid \geq a]}\right] \leq E\left[\frac{1}{a} \mid X \mid\right]$$

which is the same as Markov's inequality. In particular,

$$P[\mid X \mid \geq a] = P\left[X^2 \geq a^2\right] \leq \frac{1}{a^2} E\left[X^2\right].$$

Here assuming $E[X^2] < \infty$ and replacing X by $X - E[X]$, we get the so called Chebyshev inequality

$$P[\mid X - E[X] \mid \geq a] \leq \frac{1}{a^2} \operatorname{Var}(X) \text{ if } a > 0.$$

The distribution function of a real-valued random variable X is defined by

$$F(x) = P[X \leq x], \; x \in \mathbf{R}.$$

Furthermore, the function

$$c_X(\xi) = E\left[e^{i\xi X}\right], \; \xi \in \mathbf{R}$$

is called the characteristic function of X. Note that

$$c_X(0) = 1.$$

For example, if X is a random variable with probability distribution given by

$$P[X = 1] = P[X = -1] = \frac{1}{2}$$

then

$$c_X(\xi) = \frac{1}{2}e^{i\xi} + \frac{1}{2}e^{-i\xi} = \cos\xi.$$

Two random variables X and Y are said to have the same distribution if they have the same distribution function or, equivalently, if

$$P[X \in A] = P[Y \in A]$$

for every set A which is a finite union of intervals. Moreover, by a fairly deep theorem in Fourier analysis, this property equivalently means that the random variables X and Y have the same characteristic function.

If X is a random variable with a density function f,

$$E\left[e^{i\xi X}\right] = \int_{-\infty}^{+\infty} e^{i\xi x} f(x)dx, \; \xi \in \mathbf{R}.$$

In the special case when

$$f(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$$

we write $X \in N(0,1)$ and say that X has a standard Gaussian distribution. In this case

$$E[X] = \int_{-\infty}^{+\infty} xf(x)dx = 0$$

$$E\left[X^2\right] = \int_{-\infty}^{+\infty} x^2 f(x)dx = 1$$

and

$$\begin{aligned}\mathrm{Var}(X) &= E\left[(X - E[X])^2\right] \\ &= E\left[X^2\right] - (E[X])^2 = 1.\end{aligned}$$

A standard Gaussian random variable X has the distribution function

$$\Phi(x) = \int_{-\infty}^{x} e^{-y^2/2} \frac{dy}{\sqrt{2\pi}}, \quad x \in \mathbf{R}$$

and the characteristic function

$$c_X(\xi) = e^{-\xi^2/2}.$$

To prove the last claim first note that the Gaussian function

$$e^{-\frac{x^2}{2}}, \quad x \in \mathbf{R}$$

is even. Hence

$$\begin{aligned}c_X(\xi) &= \int_{-\infty}^{\infty} (\cos(\xi x) + i \sin(\xi x))\, e^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}} \\ &= \int_{-\infty}^{\infty} \cos(\xi x)\, e^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}}\end{aligned}$$

and we get

$$\begin{aligned}\frac{d}{d\xi} c_X(\xi) &= \int_{-\infty}^{\infty} \frac{d}{d\xi} \cos(\xi x)\, e^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}} \\ &= -\int_{-\infty}^{\infty} x \sin(\xi x)\, e^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}}.\end{aligned}$$

Now by partial integration,

$$\frac{d}{d\xi} c_X(\xi) = -\xi c_X(\xi)$$

that is,

$$\frac{d}{d\xi}(e^{\frac{\xi^2}{2}} c_X(\xi)) = 0.$$

Since $c_X(0) = 1$ it follows that

$$c_X(\xi) = e^{-\xi^2/2}.$$

Suppose $\alpha \in \mathbf{R}$ and $\sigma \geq 0$ are given real numbers. A random variable X is said to belong to the class $N(\alpha, \sigma^2)$ if

$$X = \alpha + \sigma G$$

where $G \in N(0,1)$. Moreover, in this case

$$E[X] = \alpha$$

$$\mathrm{Var}(X) = \sigma^2$$

and

$$c_X(\xi) = e^{i\alpha\xi - \frac{1}{2}\sigma^2\xi^2} = e^{i\xi E[X] - \frac{1}{2}\xi^2 \mathrm{Var}(X)}.$$

A random variable in the class $N(\alpha, \sigma^2)$ is said to have a Gaussian distribution with expectation α and variance σ^2.

A random variable X is said to have a uniform distribution in the interval $[a, b]$ if $a < b$ and X possesses the density function

$$f(x) = \begin{cases} \frac{1}{b-a} \text{ if } a \leq x \leq b \\ 0 \text{ if } x < a \text{ or } x > b. \end{cases}$$

If $X \in N(0,1)$ the random variable $\Phi(X)$ has a uniform distribution in the unit interval $[0,1]$.

Two real-valued random variables X and Y have a density f if

$$E[g(X,Y)] = \iint\limits_{\mathbf{R}^2} g(x,y) f(x,y) dx dy$$

for each function $g : \mathbf{R}^2 \to \mathbf{R}$ such that gf is integrable. Here the double integral may be evaluated by iterated integration,

$$\iint\limits_{A \times B} f(x,y) dx dy = \int_A (\int_B f(x,y) dy) dx = \int_B (\int_A f(x,y) dx) dy.$$

In particular, an event of the type $[X \in A, Y \in B]$ has the probability

$$P[X \in A, Y \in B] = \iint\limits_{A \times B} f(x,y) dx dy$$

where $A \times B = \{(x,y);\ x \in A \text{ and } y \in B\}$.

Suppose two real-valued random variables X and Y have finite second order moments. Then, for any real numbers a and b,

$$E\left[(aX+bY)^2\right] \geq 0$$

that is,

$$a^2E\left[X^2\right] + 2abE\left[XY\right] + b^2E\left[Y^2\right] \geq 0.$$

Hence

$$(aE\left[X^2\right] + bE\left[XY\right])^2 + b^2(E\left[X^2\right]E\left[Y^2\right] - E^2\left[XY\right]) \geq 0$$

and we get the so called Cauchy-Schwarz inequality,

$$\mid E\left[XY\right] \mid \leq \sqrt{E\left[X^2\right]}\sqrt{E\left[Y^2\right]}.$$

Replacing (X, Y) by $(1, \mid X \mid)$ yields

$$E\left[\mid X \mid\right] \leq \sqrt{E\left[X^2\right]}\ .$$

The covariance of two random variables X and Y with finite second order moments is defined to be

$$\mathrm{Cov}(X,Y) = E\left[(X - E\left[X\right])(Y - E\left[Y\right])\right].$$

Note that

$$\mathrm{Cov}(X,Y) = E\left[XY\right] - E\left[X\right]E\left[Y\right]$$

and

$$\mathrm{Var}(X+Y) = \mathrm{Var}(X) + 2\mathrm{Cov}(X,Y) + \mathrm{Var}(Y).$$

If, in addition,

$$\mathrm{Var}(X) > 0 \text{ and } \mathrm{Var}(Y) > 0$$

the correlation of X and Y is defined to be

$$\mathrm{Cor}(X,Y) = \frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}(X)\mathrm{Var}(Y)}}.$$

We find it convenient to define $\mathrm{Cov}(X,Y) = 0$ if either $\mathrm{Var}(X) = 0$ or $\mathrm{Var}(Y) = 0$. If $\mathrm{Cov}(X,Y) = 0$ the random variables X and Y are said to be uncorrelated and if $\mathrm{Cov}(X,Y) > 0$ (< 0) the random variables X and Y are said to be positively (negatively) correlated.

The correlation between two random variables is a measure of codependence for some distributions such as Gaussian, as shown below. However, there are uncorrelated random variables that are not independent (see the exercises in this section).

Example 3.1.1. Let U and V be random varibles and suppose $\mathrm{Var}(V) > 0$. In many applications it is important to find an $a \in \mathbf{R}$ such that $\mathrm{Var}(U - aV) \leq \mathrm{Var}(U - xV)$ for every $x \in \mathbf{R}$.

To solve this problem set $U_0 = U - E[U\ ]$ and $V_0 = V - E[V]$. We have

$$\begin{aligned} f(x) =_{def} \mathrm{Var}(U - xV) &= E\left[(U_0 - xV_0)^2\right] \\ &= E\left[U_0{}^2\right] - 2xE[U_0V_0] + x^2E\left[V_0{}^2\right] \\ &= (x\sqrt{E[V_0{}^2]} - \frac{E[U_0V_0]}{\sqrt{E[V_0{}^2]}})^2 + E\left[U_0{}^2\right] - (\frac{E[U_0V_0]}{\sqrt{E[V_0{}^2]}})^2. \end{aligned}$$

Hence

$$\min f = f(a)$$

where

$$a = \frac{E[U_0V_0]}{E[V_0{}^2]} = \frac{\mathrm{Cov}(U, V)}{\mathrm{Var}(V)}.$$

The real-valued random variables $X_1, ..., X_n$ have a joint density f if $f \geq 0$ and

$$E[g(X_1, ..., X_n)] = \underset{\mathbf{R}^n}{\int \cdots \int} g(x_1, ..., x_n) f(x_1, ..., x_n) dx_1 ... dx_n$$

for each function $g : \mathbf{R}^n \to \mathbf{R}$ such that gf is integrable, where the integration in $\mathbf{R}^n$ may be evaluated by iterated integration. In particular, an event of the type $[X_1 \in A_1, ..., x_n \in A_n]$ has the probability

$$P[X_1 \in A_1, ..., X_n \in A_n] = \underset{A_1 \times ... \times A_n}{\int \cdots \int} f(x_1, ..., x_n) dx_1 ... dx_n$$

where $A_1 \times ... \times A_n = \{(x_1, ..., x_n);\ x_1 \in A_1, ..., \ x_n \in A_n\}$. Note that

$$\underset{\mathbf{R}^n}{\int \cdots \int} f(x_1, ..., x_n) dx_1 ... dx_n = 1.$$

A collection of random variables $(X(t))_{t\in T}$ is called a stochastic process. Below we will often write X_t instead of $X(t)$. The index set T is called the time parameter set and the map

$$t \to X_t(\omega)$$

a realization, sample function, sample path, or trajectory of the process. Two stochastic processes $X = (X(t))_{t\in T}$ and $Y = (Y(t))_{t\in T}$ with the same time parameter set are said to be equivalent in distribution if, for all $t_1, ..., t_n \in T$ and $n \in \mathbf{N}_+$,

$$P\left[X(t_1) \in A_1, ..., X(t_n) \in A_n\right] = P\left[Y(t_1) \in A_1, ..., Y(t_n) \in A_n\right]$$

where $A_1, ..., A_n$ are finite unions of intervals. Measure theory tells us that this property is equivalent to

$$E\left[e^{i\sum_{k=1}^{n}\xi_k X(t_k)}\right] = E\left[e^{i\sum_{k=1}^{n}\xi_k Y(t_k)}\right]$$

for all $\xi_1, ..., \xi_n \in \mathbf{R}$, $t_1, ..., t_n \in T$, and $n \in \mathbf{N}_+$. Two stochastic processes which are equivalent in distribution are often identified.

If $X = (X(t))_{t\in T}$ and $Y = (Y(t))_{t\in T}$ are two stochastic processes with the same time set T and such that $X(t)$ and $Y(t)$ have the same distribution for every $t \in T$, the processes X and Y need not be equivalent in distribution (Exercise 14).

Suppose $(X(t))_{t\in T}$ is a stochastic process. If

$$E\left[\mid X(t) \mid\right] < \infty,\ t \in T$$

the expectation function $\alpha : T \to \mathbf{R}$ of the process is given by

$$\alpha_t = E\left[X(t)\right],\ t \in T.$$

Here the process is said to be centred if $\alpha = 0$. Moreover, if

$$E\left[X^2(t)\right] < \infty,\ t \in T$$

the covariance function $C : T \times T \to \mathbf{R}$ is defined as

$$C(s,t) = \mathrm{Cov}(X(s), X(t)),\ s,t \in T,$$

that is,

$$C(s,t) = E\left[(X(s) - \alpha_s)(X(t) - \alpha_t)\right],\ s,t \in T.$$

A stochastic process $(X_t)_{t\in\{1,\dots,n\}} = (X_k)_{k=1}^n$ is identified with an $\mathbf{R}^n$-valued random variable, that is a random vector in $\mathbf{R}^n$. If

$$E\left[\mid X_k \mid\right] < \infty,\ \ k = 1, 2, \dots, n$$

the mean value function of the process can be viewed as a vector in $\mathbf{R}^n$.

One of the most important concepts in probability is so called independence. The random variables X_k, $k = 1, \dots, n$, are said to be independent if any of the following three conditions holds:

(1)

$$P\left[X_1 \in A_1, \dots, X_n \in A_n\right] = \prod_{k=1}^{n} P\left[X_k \in A_k\right]$$

for all sets $A_1, \dots, A$, which are finite unions of intervals.

(2)

$$E\left[\prod_{k=1}^{n} g_k(X_k)\right] = \prod_{k=1}^{n} E\left[g_k(X_k)\right]$$

for all functions $g_k : \mathbf{R} \to \mathbf{C}$, only with finitely many points of discontinuity such that

$$E\left[\mid g_k(X_k) \mid\right] < \infty,\ \ ,\ k = 1, \dots, n.$$

(3)

$$E\left[e^{i\sum_{k=1}^{n} \xi_k X_k}\right] = \prod_{k=1}^{n} E\left[e^{i\xi_k X_k}\right]$$

for all $\xi_k \in \mathbf{R}$, $k = 1, \dots, n$.

To prove that (1)-(3) are equivalent falls outside the scope of this presentation.

If the random variables $X_1, \dots, X_{n-1}, X_n$ are independent and have the density functions $f_1, \dots, f_{n-1}$, and f_n, respectively, then

$$P\left[(X_1, \dots, X_n) \in A\right] = \int \cdots \int_{(x_1,\dots,x_n)\in A} \prod_{k=1}^{n} f_k(x_k) dx_1 \dots dx_n$$

for any finite union A of n-cells in $\mathbf{R}^n$. Here a subset R of $\mathbf{R}^n$ is called an n-cell if there are subintervals $I_1, \dots, I_n$ of $\mathbf{R}$ such that

$$R = \left\{x;\ x = (x_1, \dots, x_n) \in \mathbf{R}^n \text{ and } x_k \in I_k,\ k = 1, \dots, n\right\}.$$

A collection $(X(t))_{t\in T}$ of random variables is said to be independent if any finite subcollection is independent and, similarly, a collection $(A_t)_{t\in T}$ of events is said to be independent if the collection $(1_{A_t})_{t\in T}$ of random variables is independent. In particular, this means that a finite collection $A_1, ..., A_n$ of events is independent if

$$P\left[A_{k_i} \cap ... \cap A_{k_m}\right] = P\left[A_{k_i}\right] \cdot ... \cdot P\left[A_{k_m}\right]$$

whenever $2 \leq m \leq n$ and $1 \leq k_1 < ... < k_m \leq n$.

If the random variables X and Y are independent and possess finite second order moments,

$$\mathrm{Cov}(X, Y) = E\left[XY\right] - E\left[X\right] E\left[Y\right] = 0$$

that is, X and Y are uncorrelated and, in particular,

$$\mathrm{Var}(X + Y) = \mathrm{Var}(X) + \mathrm{Var}(Y).$$

A very important property of independent Gaussian random variables is that their sums are Gaussian. For example, if $X_0 \in N(\alpha_0, \sigma_0^2)$ and $X_1 \in N(\alpha_1, \sigma_1^2)$ then $X_0 + X_1 \in N(\alpha_0 + \alpha_1, \sigma_0^2 + \sigma_1^2)$. In fact,

$$E\left[e^{i\xi(X_0+X_1)}\right] = E\left[e^{i\xi X_0} e^{i\xi X_1}\right] = E\left[e^{i\xi X_0}\right] E\left[e^{i\xi X_1}\right]$$

$$= e^{i\alpha_0\xi - \frac{1}{2}\sigma_0^2\xi^2} e^{i\alpha_1\xi - \frac{1}{2}\sigma_1^2\xi^2} = e^{i(\alpha_0+\alpha_1)\xi - \frac{1}{2}(\sigma_0^2+\sigma_1^2)\xi^2}$$

and the claim follows at once. An alternative proof is as follows. For simplicity assume $\alpha_0 = \alpha_1 = 0$, $\sigma_0 > 0$ and $\sigma_1 > 0$. Then

$$P\left[X_0 + X_1 \in A\right] = \iint_{\sigma_0 y_o + \sigma_1 y_1 \in A} e^{-\frac{y_0^2}{2} - \frac{y_1^2}{2}} \frac{dy_0 dy_1}{(\sqrt{2\pi})^2}$$

and the change of variables

$$\begin{cases} z_0 = (\sigma_0 y_0 + \sigma_1 y_1)/\sqrt{\sigma_0^2 + \sigma_1^2} \\ z_1 = (\sigma_1 y_0 - \sigma_0 y_1)/\sqrt{\sigma_0^2 + \sigma_1^2} \end{cases}$$

yields

$$P\left[X_0 + X_1 \in A\right] = \iint_{z_0\sqrt{\sigma_0^2+\sigma_1^2} \in A} e^{-\frac{z_0^2}{2} - \frac{z_1^2}{2}} \frac{dz_0 dz_1}{(\sqrt{2\pi})^2}$$

$$= \int_{z_0\sqrt{\sigma_0^2+\sigma_1^2}\in A} e^{-\frac{z_0^2}{2}} \frac{dz_0}{\sqrt{2\pi}} = \int_A e^{-\frac{z^2}{2(\sigma_0^2+\sigma_1^2)}} \frac{dz}{\sqrt{2\pi(\sigma_0^2+\sigma_1^2)}}$$

and it follows that $X_0 + X_1 \in N(0, \sigma_0^2 + \sigma_1^2)$.

A stochastic process $(X(t))_{t\in T}$ is said to be Gaussian if for every $\xi_k \in \mathbf{R}$, $t_k \in T$, $k = 1, ..., n$, and $n \in \mathbf{N}_+$, the linear combination

$$Y = \sum_{k=1}^{n} \xi_k X(t_k),$$

has a Gaussian distribution. In this case,

$$Y \in N(\sum_{k=1}^{n} \xi_k E\left[X(t_k)\right], \sum_{j,k=1}^{n} \xi_j \xi_k \mathrm{Cov}(X(t_j), X(t_k)))$$

and we get

$$E\left[e^{i\sum_{k=1}^{n} \xi_k X(t_k)}\right] = e^{i\sum_{k=1}^{n} \xi_k E[X(t_k)] - \frac{1}{2}\sum_{j,k=1}^{n} \xi_j \xi_k \mathrm{Cov}(X(t_j), X(t_k))}.$$

If a Gaussian process $(X_k)_{k=1}^n$ satisfies

$$\mathrm{Cov}(X_j, X_k) = 0 \text{ if } j \neq k$$

then

$$E\left[e^{i\sum_{k=1}^{n} \xi_k X_k}\right] = e^{i\sum_{k=1}^{n} \xi_k E[X_k] - \frac{1}{2}\sum_{j,k=1}^{n} \xi_j \xi_k \mathrm{Cov}(X_j, X_k)}$$

$$= e^{i\sum_{k=1}^{n} \xi_k E[X_k] - \frac{1}{2}\sum_{k=1}^{n} \xi_k^2 \mathrm{Var}(X_k)} = \prod_{k=1}^{n} e^{i\xi_k E[X_k] - \frac{1}{2}\xi_k^2 \mathrm{Var}(X_k)}$$

$$= \prod_{k=1}^{n} E\left[e^{i\xi_k X_k}\right]$$

and it follows that $X_1, ..., X_n$ are independent. Thus we have proved the following important

Theorem 3.1.1. *Suppose* $(X_k)_{k=1}^n$ *is a Gaussian process. Then* $X_1, ..., X_n$ *are independent if and only if* $\mathrm{Cov}(X_j, X_k) = 0$ *if* $j \neq k$.

A sequence $(X_k)_{k=1}^N$ of independent identically distributed random variables is called an i.i.d. Here $N \in \mathbf{N}_+ \cup \{\infty\}$. If X is a random variable and $(X_k)_{k=1}^N$ is an i.i.d. such that X_1 and X have the same distribution, the random variables X_k, $k \geq 1$, are called independent observations of X.

If $(X_k)_{k=1}^N$ is an i.i.d. the corresponding sequence of partial sums $(Z_n)_{k=1}^N$, where $Z_n = X_1 + ... + X_n$, $n \geq 1$, is called a random walk. Moreover, if $a \in \mathbf{R}$ the process $(U_n)_{n=0}^N$, where $U_0 = a$ and $U_n = a + Z_n$, $n \geq 1$, is called a random walk which starts at the point a at the time $n = 0$. The random variables X_n, $n \geq 1$, are called increments of the random walk. The sequence $(X_n)_{n=1}^N$ is called a Gaussian i.i.d. if it is an i.i.d. with X_1 Gaussian. The corresponding random walks are called Gaussian random walks. The random walk $(Z_n)_{n=1}^N$ is called a simple random walk if $(X_k)_{k=1}^N$ is an i.i.d. and X_1 has the probability distribution given by

$$P[X_1 = 1] = P[X_1 = -1] = \frac{1}{2}.$$

For example, if $(\ln S(t))_{t=0}^T$ denotes the log-price process in the binomial model in T periods we get a random walk starting at $\ln S(0)$ at time 0.

The notions of i.i.d., independent observations, and random walk extend unambiguously to $\mathbf{R}^n$-valued random variables.

Exercises

1. Suppose the random variables $X_1, ..., X_n$ are independent and

$$E\left[X_k^2\right] < \infty,\ k = 1, ..., n.$$

   Show that

$$\mathrm{Var}(\sum_{k=1}^n X_k) = \sum_{k=1}^n \mathrm{Var}(X_k).$$

2. (Binomial model, T periods) Set

$$Y = \frac{1}{T}\sum_{t=1}^T \ln \frac{S(t)}{S(t-1)}.$$

   Prove that $E[Y] = d + p_u(u - d)$ and $\mathrm{Var}(Y) = \frac{1}{T}p_u(1 - p_u)(u - d)^2$.

3. Suppose the random variables $X_k \in N(\alpha_k, \sigma_k^2)$, $k = 1, ..., n$, are independent. Prove that $\sum_{k=1}^n X_k \in N(\sum_{k=1}^n \alpha_k, \sum_{k=1}^n \sigma_k^2)$.

4. Suppose X and Y are independent random variables with density functions f and g, respectively. Prove that $X+Y$ has the density function $f * g$, where

$$(f * g)(x) = \int_{-\infty}^{\infty} f(y)g(x-y)dy, \ x \in \mathbf{R}.$$

5. Show that $|\mathrm{Cov}(X,Y)| \leq \sqrt{\mathrm{Var}(X)}\sqrt{\mathrm{Var}(Y)}$.

6. Prove that $-1 \leq \mathrm{Cor}(X,Y) \leq 1$.

7. The random variables X and Y are independent and uniformly distributed in the unit interval $[0,1]$. Show that the random variables $\sqrt{2\ln\frac{1}{X}}\cos(2\pi Y)$ and $\sqrt{2\ln\frac{1}{X}}\sin(2\pi Y)$ are independent and $N(0,1)$-distributed.

8. Let X be a centred Gaussian random. Show that

$$E\left[e^{\xi X}\right] = e^{\frac{\xi^2}{2}E\left[X^2\right]}, \ \xi \in \mathbf{R}.$$

9. Suppose $X, Y \in N(0,1)$ are independent. Prove that

$$E\left[e^{\xi \max(X, X+Y)}\right] = e^{\xi^2}\Phi(\xi) + \frac{1}{2}e^{\frac{\xi^2}{2}}, \ \xi \in \mathbf{R}.$$

10. A random variable X has the density function $f(x) = \frac{x^2}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$, $x \in \mathbf{R}$. Find the characteristic function $c_X(\xi) = E\left[e^{i\xi X}\right]$, $\xi \in \mathbf{R}$.

11. Suppose $X \in N(\alpha, \sigma^2)$ and $K > 0$. Compute

$$E\left[\max(0, e^X - K)\right].$$

12. Prove that $\Phi(x) = 1 - \Phi(-x)$, $x \in \mathbf{R}$, and

$$\left(\frac{1}{x} - \frac{1}{x^3}\right)\frac{e^{-x^2/2}}{\sqrt{2\pi}} \leq 1 - \Phi(x) \leq \frac{1}{x}\frac{e^{-x^2/2}}{\sqrt{2\pi}}, \ x > 0.$$

13. Show that
$$1-\Phi(x) \leq \frac{1}{2}e^{-x^2/2},\ x \geq 0.$$

14. Suppose $X \in N(0,1)$ and set
$$Y = X1_{[|X|\leq c]} - X1_{[|X|>c]}$$
where c is a given positive real number. (a) Prove that $Y \in N(0,1)$. (b) First choose c such that X and Y are uncorrelated and then prove that
$$P\left[X > c, Y > c\right] = 0 \neq P\left[X > c\right] P\left[Y > c\right].$$
Explain why the random vector (X, Y) is not Gaussian.

15. Define two stochastic processes $X = (X(t))_{t\in T}$ and $Y = (Y(t))_{t\in T}$ with the same time set T such that $X(t)$ and $Y(t)$ have the same distribution for every $t \in T$ but X and Y are not equivalent in distribution. (Hint: the previous exercise.)

16. Suppose the random variable X is positive with probability one and $\ln X \in N(0,1)$. (a) Find the density function f of X. (b) Set
$$g(x) = \begin{cases} f(x)(1+\sin(2\pi \ln x)),\ x > 0, \\ 0,\ \text{if}\ \ x \leq 0. \end{cases}$$
Show that $g(x) \geq 0$ and
$$\int_0^\infty p(x)g(x)dx = \int_0^\infty p(x)f(x)dx$$
for every polynomial $p(x)$.

17. Let $x \in [0,1]$ and suppose $P\left[X = 1\right] = x$ and $P\left[X = 0\right] = 1-x$. Furthermore, let $X_1, ..., X_n$ be independent copies of X. Show that
$$E\left[f(\frac{1}{n}(X_1 + ... + X_n))\right] = \sum_{k=0}^{n} f(\frac{k}{n})\tbinom{n}{k}x^k(1-x)^{n-k}$$
for any function $f : [0,1] \to \mathbf{R}$.

18. Suppose $n \geq 2$ and $X_1, ..., X_n$ is an i.i.d. with $E[X_1] = \alpha$ and $\text{Var}(X_1) = \sigma^2$. Set

$$\bar{X} = \frac{1}{n} \sum_{k=1}^{n} X_k$$

and

$$s^2 = \frac{1}{n-1} \sum_{k=1}^{n} (X_k - \bar{X})^2.$$

Prove that

$$E\left[\bar{X}\right] = \alpha$$

and

$$E\left[s^2\right] = \sigma^2.$$

19. A random variable X is said to be Cauchy distributed with parameters $\alpha \in \mathbf{R}$ and $\beta > 0$ if

$$P[a < X < b] = \frac{1}{\pi} \int_a^b \frac{\beta dx}{\beta^2 + (x-\alpha)^2}, \text{ if } a < b$$

(abbr. $X \in C(\alpha, \beta)$).

(a) Suppose X is uniformly distributed in the interval $\left[-\frac{1}{2}, \frac{1}{2}\right]$. Show that $\tan(\pi X) \in C(0, 1)$.

(b) Suppose $X, Y \in N(0, 1)$ are independent. Prove that $\frac{Y}{X} \in C(0, 1)$.

20. The function $f : \left]a, b\right[\to \mathbf{R}$ is convex and differentiable and $b - a < \infty$. a) Show that f' is increasing and conclude that

$$f(x) \geq f(x_0) + f'(x_0)(x - x_0)$$

for every $x_0, x \in \left]a, b\right[$. b) A random variable X fulfils $a < X < b$. Prove Jensen's inequality

$$f(E[X]) \leq E[f(X)].$$

21. Suppose X is a non-negative random variable with probability density f and such that $0 < E[X^2] < \infty$. Let $\mu = E[X]$ and suppose $\alpha \in [0, 1]$.

(a) Prove that

$$\int_{\alpha\mu}^{\infty} xf(x)dx \geq (1-\alpha)\mu.$$

(b) Prove that

$$\int_{\alpha\mu}^{\infty} f(x)dx \geq (1-\alpha)^2 \frac{(E[X])^2}{E[X^2]}.$$

22. A random variable X has the density

$$f(x) = \begin{cases} \frac{1}{\pi}\sin^2 x \text{ if } \mid x \mid \leq \pi, \\ 0 \text{ otherwise.} \end{cases}$$

Find the characteristic function of X.

23. Suppose $D = \{(x,y);\ 0 < x < 1 \text{ and } y > 0\}$ and let (X,Y) be a random vector in the plane with the density function $f(x,y) = 1_D(x,y)(4x^3 + y)e^{-y}/2$. For which real t is the variance $\mathrm{Var}(X - tY)$ minimal?

24. Suppose (X,Y) is a centred Gaussian random vector in the plane. Show that

$$E\left[X^2Y^2\right] \geq E\left[X^2\right] E\left[Y^2\right].$$

3.2 The Law of Large Numbers and the Monte Carlo Method

Suppose $(X_k)_{k=1}^{\infty}$ is an i.i.d. with $E[\mid X_1 \mid] < \infty$. The Strong Law of Large Numbers says that

$$P\left[\lim_{n\to\infty} \frac{1}{n}(X_1 + ... + X_n) = E[X_1]\right] = 1.$$

Moreover, if we approximate the expectation $E[X_1]$ by the arithmetic mean $\frac{1}{n}(X_1 + ... + X_n)$ and assume that $E[X_1^2] < \infty$, the Chebyshev inequality says something about the error in the approximation

$$\frac{1}{n}(X_1 + ... + X_n) \approx E[X_1].$$

Indeed, since

$$E\left[\frac{1}{n}(X_1 + ... + X_n)\right] = E\left[X_1\right]$$

and

$$\mathrm{Var}(\frac{1}{n}(X_1 + ... + X_n)) = \frac{1}{n}\mathrm{Var}(X_1)$$

the Chebyshev inequality implies that

$$P\left[\mid \frac{1}{n}(X_1 + ... + X_n) - E\left[X_1\right] \mid \geq \varepsilon\right] \leq \frac{\mathrm{Var}(X_1)}{\varepsilon^2 n} \text{ if } \varepsilon > 0.$$

To demonstrate how this result can be used let $f : [0,1] \to \mathbf{R}$ be a continuous function and suppose we want to compute the value of the integral

$$\int_0^1 f(x)dx.$$

If a primitive F of f is known, then

$$\int_0^1 f(x)dx = F(1) - F(0).$$

In other cases, it may be useful to approximate the integral by a Riemannian sum

$$\int_0^1 f(x)dx \approx \frac{1}{n}\Sigma_{k=1}^n f(\frac{k}{n}).$$

The Law of Large Number yields a completely different approach to the problem. First we have

$$\int_0^1 f(x)dx = E\left[f(U)\right]$$

where U is a uniformly distributed random variable in the unit interval $[0,1]$. Therefore, if $(U_n)_{n=1}^{\infty}$is an i.i.d. with U_1 uniformly distributed in the unit interval

$$P\left[\int_0^1 f(x)dx = \lim_{n\to\infty} \frac{1}{n}\Sigma_{k=1}^n f(U_k)\right] = 1.$$

Using the approximation

$$\int_0^1 f(x)dx \approx \frac{1}{n}\Sigma_{k=1}^n f(U_k).$$

we say the integral is computed by the Monte Carlo method. In the case of this simple example numerical integration is preferable. However, it can be shown that the Monte Carlo method has great advantages in higher dimensions. To indicate why it is so let f be a function defined on the unit cube $Q_d = \left\{x \in \mathbf{R}^d;\ x = (x_1, ..., x_d) \text{ and } 0 \le x_k \le 1,\ k = 1, ..., n\right\}$ in $\mathbf{R}^d$. The integral

$$\int \cdots \int_{Q_d} f(x_1, ...x_d) dx_1 ... dx_d.$$

can be approximated by the Riemannian sum

$$\sum_{\substack{k_i = 0, ..., m-1 \\ i = 1, ..., d}} f(\frac{k_1}{m}, ..., \frac{k_d}{m}) \frac{1}{m^d}.$$

However, in many cases the number of terms m^d in this sum is too large to be possible to handle. Instead using the Monte Carlo method we let $U_{1k,}..., U_{dk,}\ k = 1, ..., n$ be independent and uniformly distributed random variables in the unit interval and approximate the integral by the following sum

$$\frac{1}{n} \sum_{k=1}^{n} f(U_{1k}, ..., U_{dk}).$$

This sum contains n terms and dn random numbers.

Let us return to the binomial model in T periods and consider a contingent claim paying the amount $Y = g(S(0), ..., S(T))$ to its owner at time of maturity T, where g is a deterministic function. Setting

$$f(x_1, ..., x_T) = g(S(0), S(0)e^{x_1}, ..., S(0)e^{x_1 + ... + x_T})$$

we know that the price $\Pi_Y(0)$ of the derivative at time 0 equals

$$\Pi_Y(0) = e^{-rT} E^Q \left[f(X_1, ..., X_T) \right] = e^{-rT} \sum_{x_1, ..., x_T = u \text{ or } d} f(x_1, ..., x_T) q_{x_1} \cdot ... \cdot q_{x_T}.$$

Next we introduce a new propability Q defined by the equation

$$Q\left[A\right] = E\left[\frac{q_{X_1} \cdot ... \cdot q_{X_T}}{p_{X_1} \cdot ... \cdot p_{X_T}} 1_A\right].$$

Here, if $x_1, ..., x_T = u$ or d,

$$Q\left[X_1 = x_1, ...X_T = x_T\right] = E\left[\frac{q_{X_1} \cdot ... \cdot q_{X_T}}{p_{X_1} \cdot ... \cdot p_{X_T}} 1_{[X_1=x_1,....X_T=x_T]}\right]$$
$$= \frac{q_{x_1} \cdot ... \cdot q_{x_T}}{p_{x_1} \cdot ... \cdot p_{x_T}} p_{x_1} \cdot ... \cdot p_{x_T} = q_{x_1} \cdot ... \cdot q_{x_T}$$
$$= Q\left[X_1 = x_1\right] \cdot ... \cdot Q\left[X_T = x_T\right]$$

and we conclude that the random variables $X_1, ..., X_T$ are independent with respect to the probability measure Q, which is called the martingale measure of the binomial model in T periods. To compute the price $\Pi_Y(0)$ let X_{tk}, $t = 1, ..., T$, $k = 1, ..., n$, be independent observations on X_1 relative to the probability Q. The Monte Carlo method gives us the following approximate price of the derivative,

$$\Pi_Y(0) \approx \frac{e^{-rT}}{n} \Sigma_{k=1}^n f(X_{1k}, ..., X_{Tk}).$$

Exercises

1. Use the Monte-Carlo method to find an approximate value of the integral $\int_0^1 f(x)dx$ when $a)$ $f(x) = x$ $b)$ $f(x) = \sin x$ $c)$ $f(x) = \frac{1}{x^{1/4}}$.

2. Let $f(x) = \sin(x + 1)$, $x \in \mathbf{R}$. (a) Find the value of the integral

$$I = \int_{-\infty}^{\infty} f(x) e^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}}.$$

(b) Suppose $G_1, ..., G_n \in N(0, 1)$ are independent. Find an approximate value of I using the estimates

$$MC_1 = \frac{1}{n} \sum_{k=1}^{n} f(G_k)$$

and

$$MC_2 = \frac{1}{2n} \sum_{k=1}^{n} (f(G_k) + f(-G_k))$$

respectively.

3. Suppose X is a random vector in $\mathbf{R}^d$ and B an open subset of $\mathbf{R}^d$. Set $p = P[X \in B]$ and let $X_1, ..., X_n$ be independent observations on X. Prove that
$$P[| A_n - p | \geq \varepsilon] \leq \frac{1}{4n\varepsilon^2}, \ \varepsilon > 0,$$
where
$$A_n = \frac{1}{n}(1_{[X_1 \in B]} + ... + 1_{[X_n \in B]})$$

3.3. The Central Limit Theorem

If X_n, $n \in \mathbf{N}_+$, and X are random variables such that
$$\lim_{n\to\infty} P[a < X_n < b] = P[a < X < b]$$
for all reals a and b such that
$$P[X \in \{a, b\}] = 0$$
the sequence $(X_n)_{n \in \mathbf{N}_+}$ is said to converge to X in distribution, which is denoted by
$$X_n \to X.$$
Equivalently, this type of convergence means that
$$\lim_{n\to\infty} E[f(X_n)] = E[f(X)]$$
for each bounded continuous function $f : \mathbf{R} \to \mathbf{R}$ or, alternatively
$$\lim_{n\to\infty} c_{X_n}(\xi) = c_X(\xi), \ \xi \in \mathbf{R}.$$
The proofs of these equivalences are based on measure theory and fall beyond the scope of this presentation.

Theorem 3.3.1. (Central Limit Theorem; weak form) *Let* $(X_n)_{n=1}^{\infty}$ *be an* i.i.d. *with*
$$P[X_1 = 1] = P[X_1 = -1] = \frac{1}{2}$$

and set

$$Y_n = \frac{1}{\sqrt{n}}(X_1 + ... + X_n),\ n \in \mathbf{N}_+.$$

Then

$$Y_n \to G$$

where $G \in N(0,1)$.

For stronger versions of the Central Limit Theorem see e.g. $[BOR]$. Theorem 3.3.1 was proved by de Moivre in 1733 (see e.g. the Lifshits book "Gaussian Random Functions" $[LIF]$).

PROOF OF THEOREM 3.3.1. We have

$$c_{Y_n}(\xi) = c_{X_1+...+X_n}(\frac{\xi}{\sqrt{n}}) = \prod_{k=1}^{n} c_{X_k}(\frac{\xi}{\sqrt{n}}) = \cos^n(\frac{\xi}{\sqrt{n}})$$

and therefore

$$c_{Y_n}(\xi) = (1 - \frac{\xi^2}{2n} + \frac{\xi^4}{n^2}B(\frac{\xi}{\sqrt{n}}))^n$$

where the function B is bounded in a neighbourhood of the origin. Thus

$$\lim_{n\to\infty} c_{Y_n}(\xi) = \lim_{n\to\infty} \exp(n\ln(1 - \frac{\xi^2}{2n} + \frac{\xi^4}{n^2}B(\frac{\xi}{\sqrt{n}}))$$

$$= e^{-\frac{\xi^2}{2}} = c_G(\xi)$$

which proves Theorem 3.3.1.

3.4 Problems with solutions

1. Let X be a random variable with strictly positive variance and suppose a, b, c, and d are real numbers such that $bd \neq 0$. Show that

$$\mathrm{Cor}(a + bX, c + dX) = \frac{bd}{\mid bd \mid}.$$

Solution. Set

$$U = (a + bX) - E\left[a + bX\right] = b(X - E\left[X\right])$$

and

$$V = (c + dX) - E\left[c + dX\right] = d(X - E\left[X\right]).$$

Now

$$\mathrm{Cor}(a + bX, c + dX) = \frac{\mathrm{Cov}(a + bX, c + dX)}{\sqrt{\mathrm{Var}(a + bX)}\sqrt{\mathrm{Var}(c + dX)}}$$

$$= \frac{E\left[UV\right]}{\sqrt{E\left[U^2\right]}\sqrt{E\left[V^2\right]}} = \frac{bdE\left[(X - E\left[X\right])^2\right]}{\mid b \mid\mid d \mid E\left[(X - E\left[X\right])^2\right]} = \frac{bd}{\mid bd \mid}.$$

2. (a) A random variable X has the density function

$$f(x) = \begin{cases} e^{-x}, & \text{if } x > 0, \\ 0, & \text{if } x \le 0. \end{cases}$$

Find the characteristic function c_X of X (recall that $c_X(\xi) = E\left[e^{i\xi X}\right]$ if $\xi \in \mathbf{R}$).

(b) A random variable Y has the density function

$$g(x) = \begin{cases} 0, & \text{if } x > 0, \\ e^{x}, & \text{if } x \le 0. \end{cases}$$

Find the characteristic function c_Y of Y.

(c) A random variable Z has the density function $h(x) = \frac{1}{2}e^{-|x|}$, $x \in \mathbf{R}$. Find the characteristic function c_Z of Z.

Solution. (a) For each $\xi \in \mathbf{R}$,

$$c_X(\xi) = E\left[e^{i\xi X}\right] = \int_{-\infty}^{\infty} f(x)e^{i\xi x}dx = \int_0^{\infty} e^{-x}e^{i\xi x}dx$$

$$= \int_0^{\infty} e^{x(i\xi - 1)}dx = \left[\frac{1}{i\xi - 1}e^{x(i\xi - 1)}\right]_0^{\infty}.$$

Here $\mid e^{x(i\xi-1)} \mid = \mid e^{-x}e^{i\xi x} \mid = e^{-x} \mid e^{i\xi x} \mid = e^{-x}$ and we get

$$c_X(\xi) = \frac{1}{1-i\xi}.$$

Alternatively, use that $e^{ia} = \cos a + i \sin a$ and compute

$$\int_0^\infty e^{-x}e^{i\xi x}dx = \int_0^\infty e^{-x}\cos \xi x dx + i \int_0^\infty e^{-x} \sin \xi x dx$$

by partial integration.

(b) Here $P[-Y \le y] = P[Y \ge -y] = \int_{-y}^{\infty} g(x)dx = \int_{-\infty}^{y} g(-t)dt = \int_{-\infty}^{y} f(t)dt$ and it follows that the random varibles $-Y$ and X have the same distribution. Consequently, $c_Y(\xi) = c_{-X}(\xi) = c_X(-\xi) = \frac{1}{1+i\xi}$.

(c) Since $h(x) = \frac{1}{2}f(x) + \frac{1}{2}g(x)$ and hence

$$c_Z(\xi) = \frac{1}{2}\int_{-\infty}^{\infty}(f(x)+g(x))e^{i\xi x}dx = \frac{1}{2}\left\{\frac{1}{1-i\xi} + \frac{1}{1+i\xi}\right\}$$

$$= \frac{1}{1+\xi^2}.$$

3. Suppose $\varphi(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$ and $\Phi(x) = \int_{-\infty}^{x}\varphi(t)dt$, $-\infty < x < \infty$. Prove that

$$1 - \Phi(x) \le \frac{\varphi(x)}{x}, \text{ if } x > 0,$$

and

$$1 - \Phi(x) \ge \frac{x\varphi(x)}{1+x^2}, \text{ if } x \in \mathbf{R}.$$

Solution. For any $x > 0$,

$$1 - \Phi(x) = \int_x^\infty \varphi(t)dt = \int_x^\infty \frac{1}{t}t\varphi(t)dt$$

$$\le \int_x^\infty \frac{1}{x}t\varphi(t)dt = \frac{1}{x}\left[-\varphi(t)\right]_{t=x}^{t=\infty} = \frac{\varphi(x)}{x}.$$

This proves the first inequality. To prove the second inequality define

$$f(x) = (1 + x^2)(1 - \Phi(x)) - x\varphi(x), \text{ if } x \in \mathbf{R}.$$

It is obivous that $f(x) > 0$ if $x \leq 0$ and therefore it is enough to prove that $f(x) \geq 0$ for every $x > 0$. To this end, first note that

$$\lim_{x \to \infty} (1 + x^2)(1 - \Phi(x)) = 0$$

since $0 \leq 1 - \Phi(x) \leq \frac{\varphi(x)}{x} = \frac{1}{x\sqrt{2\pi}} e^{-\frac{x^2}{2}}$ for every $x > 0$. Hence

$$\lim_{x \to \infty} f(x) = 0$$

and it is enough to show that $f'(x) \leq 0$ if $x > 0$. Now for every $x > 0$,

$$f'(x) = 2x(1 - \Phi(x)) - (1 + x^2)\varphi(x) - \varphi(x) + x^2\varphi(x)$$

$$= 2x(1 - \Phi(x) - \frac{\varphi(x)}{x}) \leq 0$$

and we are done.

CHAPTER 4

Brownian Motion

Introduction

Brownian motion is the most important stochastic process. The first applications of Brownian motion were made by Bachelier in 1900 $[BA]$ and Einstein in 1905 $[E]$. Bachelier's aim was to provide a model for option pricing and Einstein wanted to explain the physical phenomenon Brownian motion, first observed by Robert Brown $[BR]$ under his microscope in 1827 (see, Klafer, Schlesinger, Zumofen $[KSZ]$ for a very illuminating history of Brownian motion). A mathematically rigorous treatment of Brownian motion was submitted by Wiener in 1923 $[W]$.

In this chapter the Brownian motion process is introduced as a limit of scaled simple random walks. Furthermore, we show some if its connections with heat conduction and present the geometric Brownian motion model of a stock price process, which was introduced in 1965 by Samuelson $[SAM1]$ (see also $[SAM2]$).

4.1. Brownian Motion

A centred Gaussian process $(W(t))_{t\geq 0}$, starting at 0 at time 0, and with the covariance function

$$E\left[W(s)W(t)\right] = \min(s,t)$$

is called a standard Brownian motion. In thise case, $W(s)-W(t)$ is a centred Gaussian random variable with the second order moment

$$E\left[(W(s)-W(t))^2\right] = E\left[W^2(s) - 2W(s)W(t) + W^2(t)\right]$$

$$= s - 2\min(s,t) + t = \mid s-t \mid$$

and, thus

$$W(s)-W(t) \in N(0, \mid s-t \mid).$$

If $W = (W(t))_{t\geq 0}$ is a standard Brownian motion, a stochastic process $(X(t))_{t\geq 0}$ is called a Brownian motion if $X(t) = x + \sigma W(t)$, $t \geq 0$, for appropriate $x \in \mathbf{R}$ och $\sigma > 0$, and a Brownian motion with drift if $X(t) = x + \alpha t + \sigma W(t)$, $t \geq 0$, for appropriate $\alpha, x \in \mathbf{R}$ and $\sigma > 0$. Here x is called starting point, α drift constant, and σ diffusion constant.

Theorem 4.1.1. *A Gaussian process $X = (X(t))_{t\geq 0}$ is a standard Brownian motion if and only if the following conditions are true:*

(i) $X(0) = 0$

(ii) $X(t) \in N(0,t)$, $t \geq 0$

(iii) *the increments of X are independent, that is, for any finite times $0 \leq t_0 \leq t_1 \leq ... \leq t_n$ the random variables*

$$X(t_1) - X(t_0),\ X(t_2) - X(t_1), ..., X(t_n) - X(t_{n-1})$$

are independent (or uncorrelated since X is Gaussian).

PROOF First suppose $X = (X(t))_{t\geq 0}$ is a standard Brownian motion. Then (i) holds and X is a Gaussian process such that $X(t) \in N(0,t)$. This proves (ii). To prove (iii), let $j < k < n$ to get

$$E\left[(X(t_{j+1}) - X(t_j))(X(t_{k+1}) - X(t_k))\right]$$

$$= E\left[X(t_{j+1})X(t_{k+1})\right] - E\left[X(t_{j+1})X(t_k)\right] - E\left[X(t_j)X(t_{k+1})\right] + E\left[X(t_j)X(t_k)\right]$$

$$= t_{j+1} - t_{j+1} - t_j + t_j = 0.$$

This proves (iii).

Conversely, assume $X = (X(t))_{t\geq 0}$ is a Gaussian process satisfying $(i) - (iii)$. Then $X(0) = 0$ and if $0 \leq s \leq t$,

$$E\left[X(s)X(t)\right] = E\left[X(s)(X(t) - X(s)) + X^2(s)\right]$$

$$= E\left[X(s)(X(t) - X(s)\right] + E\left[X^2(s)\right]$$

$$= E[X(s)]\,E[X(t) - X(s)] + E\left[X^2(s)\right] = s.$$

From this follows that $E[X(s)X(t)] = \min(s,t)$ and X is a standard Brownian motion.

Suppose $W = (W(t))_{t\geq 0}$ is a standard Brownian motion and $a > 0$. The scaled process

$$X(t) = a^{-\frac{1}{2}}W(at),\ t \geq 0$$

is a standard Brownian motion since the process is centred, Gaussian, and

$$E[X(s)X(t)] = a^{-1}\min(as, at) = \min(s,t).$$

Furthermore,

$$Y(t) = W(t+a) - W(\,a),\ t \geq\ 0$$

is a standard Brownian motion since the process is centred, Gaussian, and

$$E[Y(s)Y(t)] = E[(W(s+a) - W(\,a))(W(t+a) - W(\,a))]$$

$$= E[(W(s+a)(W(t+a)] - E[(W(s+a)W(\,a)]$$

$$-E[(W(\,a)W(t+a)] + E(W(\,a)W(\,a)$$

$$= \min(s+a, t+a) - a - a + a = \min(s,t).$$

As a mnemonic rule we say that W starts afresh at each point of time. Finally, the sign changed process

$$Z(t) = -W(t),\ t \geq 0$$

is a standard Bownian motion..

To show the existence of Brownian motion (in the mathematical sence of the concept) requires lots of prerequistes in mathematics and we can not go into the details here. Instead the approach below is very intuitive and the motion will be the formal limit case of scaled simple random walks.

Suppose $X = (X_n)_{n=1}^{\infty}$ is an i.i.d. with

$$P[X_1 = 1] = P[X_1 = -1] = \frac{1}{2}$$

(the existence of X is equivalent to the existence of the so called Lebesgue measure and is far from trivial). We have

$$E[X_1] = 0$$

and

$$\mathrm{Var}(X_1) = 1.$$

Set

$$U_n = \sum_{1 \le k \le n} X_k,\ n \in \mathbf{N}$$

with the convention $U_0 = 0$. Thus $U = (U_n)_{n=0}^{\infty}$ is a simple random walk starting at the origin at time 0. Now if t is a real number, $[t]$ denotes the greatest integer smaller than or equal to t, and we introduce the process

$$Y(t) = U_{[t]} + (t - [t])X_{[t]+1},\ t \ge 0$$

which, in particular, equals U_n if $t = n$ and is an affine function in each time interval $[n, n+1]$, $n \in \mathbf{N}$. Moreover, the process $Y(t)$, $t \ge 0$, has continuous sample paths. Next let $N \in \mathbf{N}_+$ be fixed and set

$$W_N(t) = \frac{1}{\sqrt{N}} Y(Nt),\ t \ge 0.$$

The process W_N is a centred process and choosing $t = \frac{n}{N}$, where n is a natural number, gives

$$W_N(\frac{n}{N}) = \frac{1}{\sqrt{N}} U_n = \frac{1}{\sqrt{N}} \sum_{1 \le k \le n} X_k,\ \ n \in \mathbf{N}.$$

If N is large and $\frac{n}{N}$ fixed, by the Central Limit Theorem, the random variable

$$W_N(\frac{n}{N}) = \sqrt{\frac{n}{N}} \frac{1}{\sqrt{n}} \sum_{1 \le k \le n} X_k$$

is approximately Gaussian distributed and, in addition,

$$E\left[W_N(\frac{m}{N}) W_N(\frac{n}{N})\right] = \frac{1}{N} E\left[\sum_{k=1}^{m} X_k \sum_{k=1}^{n} X_k\right]$$

$$= \frac{1}{N} \min(m, n) = \min(\frac{m}{N}, \frac{n}{N}).$$

The process $W_N = (W_N(t))_{t \ge 0}$ has continuous sample paths and these are affine functions in each interval $\left[\frac{n}{N}, \frac{n+1}{N}\right]$, $n = 0, 1, 2, ...$. For large N, W_N approximates standard Brownian motion very well. The proof is omitted here. In the following it is often useful to think of Brownian motion as "a simple random walk in continuous time".

Theorem 4.1.2. **(Wiener's Theorem)** *There is a standard Brownian motion possessing continuous sample paths.*

One of the simplest proofs of Wiener's Theorem is given in Bass $[BASS]$. In the following, if not otherwise stated, $W = (W(t))_{t\geq 0}$ will always denote a standard Brownian motion with continuous sample paths.

Exercises

1. Define $X(0) = 0$ and $X(t) = tW(\frac{1}{t})$, $t > 0$. Show that $(X(t))_{t\geq 0}$ is a standard Brownian motion.

2. Suppose the process $(V_n(t))_{0\leq t\leq 1}$, has continuous sample paths, which are affine in each subinterval $\frac{k-1}{n} \leq t \leq \frac{k}{n}$, $k = 1, ..., n$. Moreover, assume $V_n(0) = 0$ and

$$V_n(\frac{k}{n}) = \frac{1}{\sqrt{n}} \sum_{j=1}^{k} X_j, \; k = 1, ..., n$$

where $(X_j)_{j=1}^{n}$ is an i.i.d. Draw a picture of at least two realizations of the process $(V_n(t))_{0\leq t\leq 1}$, if

a) $X_1 \in N(0,1)$.

b) X_1 is Cauchy distributed with parameters 0 and 1, that is

$$P[X_1 \in A] = \frac{1}{\pi} \int_A \frac{dx}{1+x^2}$$

(hint: represent X_1 as $\tan(\pi U)$, where U has a uniform distribution in $]-\frac{1}{2}, \frac{1}{2}[$).

4.2. The Geometric Brownian Motion Model of a Stock Price

A stock price process $S = (S(t))_{t\geq 0}$ is called a geometric Brownian motion if the so called log-price process

$$\ln S(t),\, t \geq 0$$

is governed by a Brownian motion with drift, or stated otherwise,

$$S(t) = S(0)e^{\alpha t + \sigma W(t)},\ t \geq 0$$

for appropriate parameters $\alpha \in \mathbf{R}$ och $\sigma > 0$. Since

$$E\left[S(t)\right] = S(0)e^{(\alpha + \frac{\sigma^2}{2})t}$$

it is natural to introduce a new parameter μ defined by the equation

$$\mu = \alpha + \frac{\sigma^2}{2}$$

so that

$$S(t) = S(0)e^{(\mu - \frac{\sigma^2}{2})t + \sigma W(t)}.$$

The parameters μ and σ are called the mean rate of return and the volatility of S, respectively. If the time unit is years and $\sigma = 0.25$, S is said to have the volatility 25 %. The geometric Brownian motion model of a stock price was introduced in 1965 by Samuelson $[SAM1]$. Bachelier $[BA]$ in 1900 used Brownian motion (not geometric Brownian motion) as a model of stock prices, even if Brownian motion can take negative values.

In the geometric Brownian motion model, for any $h > 0$, the sequence of log-prices

$$(\ln S(nh))_{n=0}^{\infty}$$

is a Gaussian random walk. Moreover, if the time scale is changed so that

$$\hat{t} = ct$$

where $c > 0$, the stock price process is a geometric Brownian motion in the new time unit. To explain this let

$$\hat{S}(\hat{t}) = S(t)$$

or, what amounts to the same thing,

$$\hat{S}(\hat{t}) = S(0)e^{\frac{\alpha}{c}\hat{t}+\sigma W(\frac{\hat{t}}{c})}.$$

Now setting

$$\hat{W}(\hat{t}) = \sqrt{c}W(\frac{\hat{t}}{c}),\ \hat{t} \geq 0$$

we get a a new standard Brownian motion and

$$\hat{S}(\hat{t}) = \hat{S}(0)e^{\hat{\alpha}\hat{t}+\hat{\sigma}\hat{W}(\hat{t})}$$

where

$$\hat{\alpha} = \frac{\alpha}{c} \text{ and } \hat{\sigma} = \frac{\sigma}{\sqrt{c}}.$$

Hence, the process $(\hat{S}(\hat{t}))_{\hat{t}\geq 0}$ is a geometric Brownian motion.

Theorem 4.2.1. *Let*

$$S(t) = S(0)e^{\alpha t+\sigma W(t)},\ t \geq 0$$

and suppose $0 < t_1 < ... < t_n$ *and* $a_1 < b_1, ..., a_n < b_n$. *Then*

$$P\left[a_1 < S(t_1) < b_1, ..., a_n < S(t_n) < b_n\right]$$

$$= \int_{A_1\times...\times A_n}\!\!\cdots\int \prod_{k=1}^{n}\left\{\frac{1}{\sqrt{2\pi(t_k - t_{k-1})}}e^{-\frac{(x_k-x_{k-1})^2}{2(t_k-t_{k-1})}}\right\}dx_1...dx_n.$$

where $x_0 = 0$, $t_0 = 0$, *and*

$$A_k = \left]\frac{1}{\sigma}(\ln\frac{a_k}{S(0)} - \alpha t_k), \frac{1}{\sigma}(\ln\frac{b_k}{S(0)} - \alpha t_k)\right[,\ k = 1, ..., n.$$

PROOF We have

$$P\left[a_1 < S(t_1) < b_1, ..., a_n < S(t_n) < b_n\right]$$

$$= P\left[W(t_1) \in A_1, ..., W(t_n) \in A_n\right].$$

Furthermore, defining

$$Y_1 = W(t_1),\ Y_2 = W(t_2) - W(t_1), ..., Y_n = W(t_n) - W(t_{n-1})$$

the random variables $Y_1, ..., Y_n$ are independent with Gaussian distributions

$$Y_k \in N(0, t_k - t_{k-1}),\ k = 1, ..., n$$

and

$$P\left[W(t_1) \in A_1, ..., W(t_n) \in A_n\right].$$

$$= P\left[Y_1 \in A_1, Y_1 + Y_2 \in A_2, ..., Y_1 + Y_2 + ... + Y_n \in A_n\right]$$

$$= \int \cdots \int_{y_1 \in A_1, ..., y_1 + ... + y_n \in A_n} \prod_{k=1}^{n} \left\{ \frac{1}{\sqrt{2\pi(t_k - t_{k-1})}} e^{-\frac{y_k^2}{2(t_k - t_{k-1})}} \right\} dy_1 ... dy_n$$

$$= \underset{A_1 \times ... \times A_n}{\int \cdots \int} \prod_{k=1}^{n} \left\{ \frac{1}{\sqrt{2\pi(t_k - t_{k-1})}} e^{-\frac{(x_k - x_{k-1})^2}{2(t_k - t_{k-1})}} \right\} dx_1 ... dx_n.$$

This proves Theorem 4.2.1.

Next we will discuss some statistical estimates of the parameters α and σ in the geometric Brownian motion model. To this end fix a period of time from 0 to T and choose a natural number n. Set $h = T/n$, $t_i = ih$, $i = 1, ..., n$, and

$$X_i = \ln \frac{S(t_i)}{S(t_{i-1})} = \alpha h + \sigma \sqrt{h} G_i,$$

for $i = 1, ..., n$, where $G_1, ..., G_n \in N(0, 1)$ are independent. Furthermore, define

$$\hat{\alpha} = \frac{1}{T} \Sigma_{i=1}^{n} X_i$$

and

$$\hat{\sigma}^2 = \frac{1}{T} \Sigma_{i=1}^{n} X_i^2.$$

It is readily seen that

$$\begin{cases} E\left[\hat{\alpha}\right] = \alpha \\ \mathrm{Var}(\hat{\alpha}) = \sigma^2/T \end{cases}$$

and after some calculations the following formulas are obtained, viz.

$$\begin{cases} E\left[\hat{\sigma}^2\right] = \sigma^2 + \alpha^2 T/n \\ \mathrm{Var}(\hat{\sigma}^2) = 2\sigma^4/n + 4\alpha^2 T \sigma^2/n^2. \end{cases}$$

Thus $E\left[\hat{\sigma}^2\right] - \sigma^2$ and $\mathrm{Var}(\hat{\sigma}^2)$ both converge to zero as n tends to infinity. Here note T can be an arbitrarily small positive number but in spite of this $\hat{\sigma}^2$ becomes an almost unbiased estimator of σ^2 with small variance by increasing the number of observations. In contrast to this the variance of the estimator $\hat{\alpha}$ does not change at all by increasing n. Note that $\hat{\alpha} - \alpha \in N(0, \sigma^2/T)$ so that

$$P\left[\mid \hat{\alpha} - \alpha \mid \leq \sigma x/\sqrt{T}\right] = 2\Phi(x) - 1.$$

The choice $x = 1.96$ gives $2\Phi(x) - 1 = 0.95$. If, in addition, the annual volatility is 30%, that is $\sigma = 0.3$, then $\sigma x/\sqrt{T} = 0.02$ if $T \approx 864$ years! In addition, it can be proved that $\hat{\alpha}$ is an unbiased estimator of α with smallest variance. Accordingly from this example, we conclude that the parameter α cannot be estimated by sufficiently small variance.

To explain why the parameter α is important in portfolio theory suppose the initial wealth 1 is invested in the stock and bond. Moreover, let the amount x be invested in the stock and the remaining wealth $1 - x$ in the bond so that at time T the wealth equals

$$V(x) = xe^{\alpha T + \sigma W(T)} + (1 - x)e^{rT}.$$

Here

$$E\left[V(x)\right] = xe^{\mu T} + (1 - x)e^{rT}$$

and

$$\mathrm{Var}(V(x)) = \mathrm{Var}(xe^{\alpha T + \sigma W(T)}) = x^2 e^{2\mu T}(e^{\sigma^2 T} - 1).$$

To obtain optimal wealth we maximize expected terminal wealth plus the variance of terminal wealth multiplied by a negative constant. Therefore, let

$$f(x) = E\left[V(x)\right] - \frac{1-\gamma}{2}\mathrm{Var}(V(x))$$

where $\gamma \in \left]-\infty, 1\right[$ is a parameter called the investor's risk aversion. A simple calculation shows that the maximum of f is attained at the point

$$x(T) = \frac{1}{1-\gamma}\frac{e^{\mu T} - e^{rT}}{e^{2\mu T}(e^{\sigma^2 T} - 1)}.$$

As $T \to 0$ we get

$$x(0+) = \frac{1}{1-\gamma}\frac{\mu - r}{\sigma^2}.$$

Here μ, like α, is very difficult to estimate. It is a great success in mathematical finance that the pricing and hedging of options are independent of α as will be seen below.

Exercises

1. Consider a stock price process $(S(t))_{t\geq 0}$ in the geometric Brownian motion model and introduce

$$X_k = \ln \frac{S(k)}{S(k-1)}, \; k = 1, ..., n,$$

$$\bar{X} = \frac{1}{n} \sum_{k=1}^{n} X_k$$

and

$$s^2 = \frac{1}{n-1} \sum_{k=1}^{n} (X_k - \bar{X})^2.$$

Prove that

$$E\left[\bar{X}\right] = (\text{mean rate of return}) - \frac{1}{2}(\text{volatility})^2$$

and

$$\sqrt{E\left[s^2\right]} = \text{volatility}.$$

4.3. Brownian Motion and Heat Conduction

In what follows $\mathcal{E}_c$ denotes the class of all real-valued continuous functions f on $\mathbf{R}$ such that

$$\mid f(x) \mid \leq Ae^{B|x|}, \; x \in \mathbf{R}$$

for appropriate constants $A, B > 0$, possibly dependent on f.

Now suppose the function $f \in \mathcal{E}_c$ is given and consider the heat equation

$$\frac{\partial u}{\partial \tau} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2}, \;\; \tau > 0, \;\; x \in \mathbf{R}$$

with the initial condition

$$u_{|\tau=0} = f.$$

The so called heat kernel

$$\gamma(\tau, x) = \frac{1}{\sqrt{2\pi\tau}} e^{-\frac{x^2}{2\tau}}, \;\; \tau > 0, \;\; x \in \mathbf{R}$$

solves the heat equation since

$$\frac{\partial \gamma}{\partial \tau} = -\frac{1}{2}\frac{1}{\tau^{\frac{3}{2}}}\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2\tau}} + \frac{x^2}{2\tau^{\frac{5}{2}}}\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2\tau}},$$

$$\frac{\partial \gamma}{\partial x} = -\frac{x}{\tau^{\frac{3}{2}}}\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2\tau}},$$

and

$$\frac{\partial^2 \gamma}{\partial x^2} = -\frac{1}{\tau^{\frac{3}{2}}}\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2\tau}} + \frac{x^2}{\tau^{\frac{5}{2}}}\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2\tau}}.$$

Thus

$$\frac{\partial \gamma}{\partial \tau} = \frac{1}{2}\frac{\partial^2 \gamma}{\partial x^2}.$$

In the next step define

$$u(\tau, x) = \int_{-\infty}^{\infty} f(y)\gamma(\tau, x - y)dy, \;\; \tau > 0, \;\; x \in \mathbf{R}$$

and by interchanging the order of differentiation and integration we get

$$\frac{\partial u}{\partial \tau} - \frac{1}{2}\frac{\partial^2 u}{\partial x^2} = \int_{-\infty}^{\infty} f(y)(\frac{\partial}{\partial \tau}\gamma(\tau, x - y) - \frac{1}{2}\frac{\partial^2}{\partial x^2}\gamma(\tau, x - y))dy \;\; = 0$$

and it follows that u is a solution of the heat equation in the domian $\tau > 0$, $x \in \mathbf{R}$. To check the limit of $u(\tau, x)$ ia τ goes to zero, first note that

$$u(\tau, x) = \int_{-\infty}^{\infty} f(x - y)\gamma(\tau, y)dy$$

$$= \int_{-\infty}^{\infty} f(x-y) \frac{1}{\sqrt{2\pi\tau}} e^{-\frac{y^2}{2\tau}} dy$$

and

$$u(\tau, x) = \int_{-\infty}^{\infty} f(x - \sqrt{\tau} y) \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy.$$

Hence,

$$\lim_{\tau \to 0} u(\tau, x) = \int_{-\infty}^{\infty} \lim_{\tau \to 0} f(x - \sqrt{\tau} y) \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy = \int_{-\infty}^{\infty} f(x) \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy$$

$$= f(x) \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy = f(x)$$

as

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy = 1.$$

In the following

$$u(\tau, x) = \int_{-\infty}^{\infty} f(x - \sqrt{\tau} y) \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy$$

is called the solution of the following heat conduction problem

$$\begin{cases} \frac{\partial u}{\partial \tau} = \frac{1}{2} \frac{\partial^2 u}{\partial x^2} \\ u_{|\tau=0} = f, \ \tau > 0, \ x \in \mathbf{R}. \end{cases}$$

Actually, there are more solutions to this problem but they are of no interest here.

A function $f : \mathbf{R} \to \mathbf{R}$ is said to belong to the class $\mathcal{E}$ if there exist finitely many real numbers $a_1 < ... < a_n$ such that f restricted to the set $\mathbf{R} \backslash \{a_1, ..., a_n\}$ is continuous, the limits

$$\lim_{x \nearrow a_k} f(x) \text{ and } \lim_{x \searrow a_k} f(x)$$

exist and are real numbers for $k = 1, ..., n$, and, furthermore,

$$| f(x) | \leq A e^{B|x|}, \ x \in \mathbf{R}$$

for appropriate constants $A, B > 0$. If $f \in \mathcal{E}$, the function

$$u(\tau, x) = \int_{-\infty}^{\infty} f(x - \sqrt{\tau} y) \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy$$

$$= \int_{-\infty}^{\infty} f(x+\sqrt{\tau}y)\frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}dy$$

is defined to be the solution of the heat conduction problem

$$\begin{cases} \frac{\partial u}{\partial \tau} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} \\ u_{|\tau=0} = f, \ \tau > 0, \ x \in \mathbf{R}. \end{cases}$$

Clearly, if $G \in N(0,1)$,

$$u(\tau, x) = E\left[f(x+\sqrt{\tau}G)\right].$$

or

$$u(\tau, x) = E\left[f(x+W(\tau))\right]$$

since $W(\tau)$ and $\sqrt{\tau}G$ possess the same probability distribution. From now on, if not otherwise stated, G will always denote a standard Gaussian random variable.

Example 4.3.1. Suppose

$$\begin{cases} \frac{\partial u}{\partial \tau} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} \\ u_{|\tau=0} = f, \ \tau > 0, \ x \in \mathbf{R} \end{cases}$$

where

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{x^2}{2\sigma^2}}$$

and $\sigma > 0$. The solution equals

$$u(\tau, x) = \int_{\mathbf{R}} \frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{(x-y)^2}{2\sigma^2}}\frac{1}{\sqrt{2\pi\tau}}e^{-\frac{y^2}{2\tau}}dy$$

and a calculation yields

$$u(\tau, x) = \frac{1}{\sqrt{2\pi(\sigma^2+\tau)}}e^{-\frac{x^2}{2(\sigma^2+\tau)}}$$

(note that $u(\tau, x)$ is the density function of the random variable $\sigma G + \sqrt{\tau}H$, where $G, H \in N(0,1)$ are independent).

Next consider the terminal problem

$$\begin{cases} \frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial^2 u}{\partial x^2} = 0 \\ u_{|t=T} = f,\ 0 \leq\ t < T, x \in \mathbf{R} \end{cases}$$

where T is a strictly positive real number and $f \in \mathcal{E}$. By setting

$$\tau = T - t$$

and

$$u(t,x) = v(\tau, x)$$

we get

$$\begin{cases} \frac{\partial v}{\partial \tau} = \frac{1}{2}\frac{\partial^2 v}{\partial x^2} \\ v_{|\tau=0} = f,\ \ 0 < \tau \leq T,\ \ x \in \mathbf{R}. \end{cases}$$

Hence

$$u(t,x) = v(\tau,x) = E\left[f(x + W(\tau))\right].$$

Theorem 4.3.1 *Suppose* $a, b \in \mathbf{R}$ *and* $\sigma > 0$. *If* $f \in \mathcal{E}$, *the equation*

$$\begin{cases} \frac{\partial u}{\partial t} + \frac{\sigma^2}{2}\frac{\partial^2 u}{\partial x^2} + a\frac{\partial u}{\partial x} + bu = 0 \\ u_{|t=T} = f,\ 0 \leq\ t < T, x \in \mathbf{R} \end{cases}$$

has the solution

$$\begin{aligned} u(t,x) &= e^{b\tau}E\left[f(x + a\tau + \sigma W(\tau))\right] \\ &= e^{b\tau}E\left[f(x + a\tau + \sigma\sqrt{\tau}G)\right]. \end{aligned}$$

PROOF Set

$$y = (x + a\tau)/\sigma$$

and

$$u(t,x) = e^{b\tau}v(\tau, y).$$

A straight-forward calculation yields

$$\frac{\partial v}{\partial \tau} = \frac{1}{2}\frac{\partial^2 v}{\partial y^2}$$

and

$$v_{|\tau=0} = f(\sigma y).$$

Hence

$$v(\tau, y) = E\left[f(\sigma(y + W(\tau)))\right] = E\left[f(x + a\tau + \sigma W(\tau))\right]$$

which completes the proof of Theorem 4.3.1.

Theorem 4.3.2. *Suppose $a, b \in \mathbf{R}$ and $\sigma > 0$. If $g(e^x) \in \mathcal{E}$, the equation*

$$\begin{cases} \frac{\partial u}{\partial t} + \frac{\sigma^2 s^2}{2}\frac{\partial^2 u}{\partial s^2} + as\frac{\partial u}{\partial s} + bu = 0 \\ u_{|t=T} = g,\ 0 \leq t < T,\ s > 0 \end{cases}$$

has the solution

$$\begin{aligned} u(t, s) &= e^{b\tau} E\left[g(se^{(a-\frac{\sigma^2}{2})\tau + \sigma W(\tau)})\right] \\ &= e^{b\tau} E\left[g(se^{(a-\frac{\sigma^2}{2})\tau + \sigma\sqrt{\tau}G})\right]. \end{aligned}$$

PROOF Set

$$s = e^x$$

and

$$u(t, s) = v(t, x).$$

Then

$$\frac{\partial u}{\partial s} = \frac{\partial v}{\partial x}\frac{1}{s}$$

and

$$\frac{\partial^2 u}{\partial s^2} = \frac{\partial^2 v}{\partial x^2}\frac{1}{s^2} - \frac{\partial v}{\partial x}\frac{1}{s^2}.$$

By using these formulas, the differential equation in Theorem 4.3.2 equals

$$\frac{\partial v}{\partial t} + \frac{\sigma^2}{2}\frac{\partial^2 v}{\partial x^2} + (a - \frac{\sigma^2}{2})\frac{\partial v}{\partial x} + bv = 0\ .$$

Finally observing that

$$v(T, x) = g(e^x)$$

Theorem 4.3.1 yields

$$u(t, s) = v(t, x) = e^{b\tau} E\left[g(e^{x + (a-\frac{\sigma^2}{2})\tau + \sigma W(\tau)})\right]$$

and Theorem 4.3.2 follows as $s = e^x$.

Exercises

1. Consider the equation

$$\frac{\partial u}{\partial t} + \frac{\sigma^2}{2}\frac{\partial^2 u}{\partial x^2} + a\frac{\partial u}{\partial x} + bu = 0$$

where $a, b \in \mathbf{R}$ and $\sigma > 0$. Find $\alpha, \beta \in \mathbf{R}$ so that the substitution

$$u(t,x) = e^{\alpha t + \beta x} v(t,x)$$

leads to the simpler equation

$$\frac{\partial v}{\partial t} + \frac{\sigma^2}{2}\frac{\partial^2 v}{\partial x^2} = 0\ .$$

2. Solve the equation

$$\begin{cases} \frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial^2 u}{\partial x^2} = 0 \\ u(T,x) = \max(0,x),\ 0 \le t < T,\ x \in \mathbf{R}. \end{cases}$$

(Answer: $u(t,x) = x\Phi(\frac{x}{\sqrt{\tau}}) + \sqrt{\tau}\varphi(\frac{x}{\sqrt{\tau}})$, where $\varphi = \Phi'$)

3. Suppose $f(s) = 0$ if $0 < s \le 1$ and $f(s) = 1$ if $s > 1$. Solve

$$\begin{cases} \frac{\partial u}{\partial t} + \frac{\sigma^2 s^2}{2}\frac{\partial^2 u}{\partial s^2} + s\frac{\partial u}{\partial s} - u = 0 \\ u(T,s) = f(s),\ 0 \le t < T,\ s > 0. \end{cases}$$

4. Solve the equation

$$\begin{cases} \frac{\partial u}{\partial t} + \frac{\sigma^2 s^2}{2}\frac{\partial^2 u}{\partial s^2} + s\frac{\partial u}{\partial s} - u = 0 \\ u(T,s) = s,\ 0 \le t < T,\ s > 0. \end{cases}$$

(Answer: $u(t,s) = s$)

5. Solve the initial problem

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} \\ u(0,x) = \sin x, \;\; t > 0, \; x \in \mathbf{R} \end{cases}$$

Find an approximation of $u(4,1)$ by using the Monte-Carlo method.
(Answer: $u(t,s) = e^{-t/2}\sin x$)

4.4. Simple Random Walk and a Numerical Method for the Heat Equation

If $f \in \mathcal{E}$, Theorem 4.3.1 proves that the equation

$$\begin{cases} \frac{\partial u}{\partial t} + \frac{\sigma^2}{2}\frac{\partial^2 u}{\partial x^2} = 0 \\ u_{|t=T} = f, \; 0 \leq \; t < T, x \in \mathbf{R} \end{cases}$$

has the solution

$$u(t,x) = E\left[f(x + \sigma W(\tau))\right]$$

where $\tau = T - t$. Now suppose $(\Sigma_{1 \leq k \leq n} X_k)_{n=0}^{\infty}$ is a simple random walk, set

$$h = \tau/N$$

and note that, if N is big, the random variable

$$\sqrt{h}\sum_{j=1}^{N} X_j$$

is approximately $N(0,\tau)$-distributed by the Central Limit Theorem. Thus, since $W(\tau) \in N(0,\tau)$, the function

$$v(t,x) = E\left[f(x + \sigma\sqrt{h}\sum_{j=1}^{N} X_j\right]$$

approximates $u(t,x)$. Below it will be proved that a certain numerical method for the heat equation leads to the same formula.

Consider the equation

$$\frac{\partial u}{\partial t} + \frac{\sigma^2}{2}\frac{\partial^2 u}{\partial x^2} = 0$$

and use the following approximations

$$\frac{\partial u}{\partial t} \approx \frac{u(t,x) - u(t - \Delta t, x)}{\Delta t},$$

$$\frac{\partial u}{\partial x}(t,x) \approx \frac{u(t,x) - u(t - \Delta x, x)}{\Delta x},$$

and

$$\frac{\partial^2 u}{\partial x^2}(t,x) \approx \frac{u(t, x + \Delta x) - 2u(t,x) + u(t, x - \Delta x)}{(\Delta x)^2}$$

and replace u by v to obtain

$$\frac{v(t,x) - v(t - \Delta t, x)}{\Delta t}$$
$$+ \frac{\sigma^2}{2} \frac{v(t, x + \Delta x) - 2v(t,x) + v(t, x - \Delta x)}{(\Delta x)^2} = 0.$$

To simplify, let

$$\kappa = \frac{\sigma^2}{2} \frac{\Delta t}{(\Delta x)^2}$$

so that

$$v(t,x) - v(t - \Delta t, x) + \kappa(v(t, x + \Delta x) - 2v(t,x) + v(t, x - \Delta x)) = 0$$

or

$$v(t - \Delta t, x) = \kappa v(t, x + \Delta x) + (1 - 2\kappa)v(t,x) + \kappa v(t, x - \Delta x).$$

This equation opens for probabilistic interpretations for each $\kappa \in \left]0, \frac{1}{2}\right]$. Here we concentrate on the special case $\kappa = \frac{1}{2}$ and let

$$\Delta t = h \text{ and } \Delta x = \sigma\sqrt{h}$$

Then

$$v(t - h, x) = \frac{1}{2} v(t, x + \sigma\sqrt{h}) + \frac{1}{2} v(t, x - \sigma\sqrt{h})$$

or

$$v(t,x) = \frac{1}{2} v(t + h, x + \sigma\sqrt{h}) + \frac{1}{2} v(t + h, x - \sigma\sqrt{h})$$

that is,

$$v(t,x) = E\left[v(t + h, x + \sigma\sqrt{h}X_1)\right].$$

Since X_1 and X_2 are independent,

$$v(t,x) = E\left[v(t+2h, x+\sigma\sqrt{h}(X_1+X_2))\right]$$

and by repetition we get

$$v(t,x) = E\left[v(T, x+\sigma\sqrt{h}\sum_{j=1}^{N} X_j)\right].$$

The terminal condition

$$v(T,x) = f(x)$$

now yields

$$v(t,x) = E\left[f(x+\sigma\sqrt{h}\sum_{j=1}^{N} X_j)\right]$$

which is the same formula as the one obtained above by approximating Brownian motion with a simple random walk.

4.5 Problems with solutions

1. Set $X(t) = W(t) - tW(1)$ and $Y(t) = X(1-t)$ if $0 \le t \le 1$. Prove that the processes $(X(t))_{0\le t\le 1}$ and $(Y(t))_{0\le t\le 1}$ are equivalent in distribution. (Hint: Prove that the processes are Gaussian with $E[X(t)] = E[Y(t)]$ and $\mathrm{Cov}[X(s),X(t)] = \mathrm{Cov}(Y(s),Y(t))$ for all $0 \le s,t \le 1$.)

Solution. Given $t_1,...,t_n \in [0,1]$ an arbitrary linear combination of $X(t_1),...,X(t_n)$ is a linear combination of $W(t_1),...,W(t_n),W(1)$ and, hence a centred Gaussian random variable. In a similar way a linear combination of $Y(t_1),...,Y(t_n)$ is a centred Gaussian random variable. Therefore it only remains to prove that the processes $(X(t))_{0\le t\le 1}$ and $(Y(t))_{0\le t\le 1}$ have the same covariance. To this end let $0 \le s \le t \le 1$. Then

$$E[X(s)X(t)] = E[(W(s)-sW(1))(W(t)-tW(1)]$$

$$= E\left[W(s)W(t)\right] - tE\left[W(s)W(1)\right] - sE\left[W(1)W(t)\right] + stE\left[(W^2(1)\right]$$
$$= s - st - st + st = s - st$$

and

$$E\left[Y(s)Y(t)\right] = E\left[X(1-t)X(1-s)\right] = (1-t) - (1-t)(1-s) = s - st.$$

Thus $E\left[X(s)X(t)\right] = E\left[Y(s)Y(t)\right] = \min(s,t) - st$ for all $0 \leq s, t \leq 1$ and it follows that the processes $(X(t))_{0\leq t\leq 1}$ and $(Y(t))_{0\leq t\leq 1}$ are equivalent in distribution.

2. Suppose W denotes a standard Brownian motion. Find

$$E\left[(W(t) + W^2(t))e^{W^2(t)}\right]$$

for every $t \in [0, 1/2[$.

Solution. We have

$$E\left[(W(t) + W^2(t))e^{W^2(t)}\right] = \int_{\mathbf{R}} \left(\sqrt{t}x + (\sqrt{t}x)^2\right) e^{(\sqrt{t}x)^2} e^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}}$$
$$= t \int_{\mathbf{R}} x^2 e^{tx^2} e^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}} = t \int_{\mathbf{R}} x^2 e^{-\frac{1}{2}(1-2t)x^2} \frac{dx}{\sqrt{2\pi}}$$
$$= \left\{y = \sqrt{1-2t}x\right\} = \frac{t}{(1-2t)^{\frac{3}{2}}} \int_{\mathbf{R}} y^2 e^{-\frac{y^2}{2}} \frac{dy}{\sqrt{2\pi}}$$
$$= \frac{t}{(1-2t)^{\frac{3}{2}}}.$$

CHAPTER 5

The Black-Scholes Option Theory

Introduction

In this chapter we consider a capital market consisting of a stock, a bond and options on the stock. It will be assumed that the stock price process $(S(t))_{t\geq 0}$ is governed by a geometric Brownian motion with mean rate of return μ and volatility σ, that is

$$S(t) = S(0)e^{\alpha t+\sigma W(t)}$$

where $\alpha = \mu - \frac{\sigma^2}{2}$. Furthermore the bond price process $(B(t))_{t\geq 0}$ is given by

$$B(t) = B(0)e^{rt}$$

where $r > 0$ is a constant, called interest rate. At time 0, the asset prices $S(0)$ and $B(0)$ are known strictly positive real numbers.

In the seminal paper $[BS]$, which appeared in 1973, Black and Scholes derived the following price at time 0 of a European call with strike price K and termination date T, namely

$$c(0, S(0), K, T)$$

$$= S(0)\Phi(\frac{\ln \frac{S(0)}{K} + (r + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}) - Ke^{-rT}\Phi(\frac{\ln \frac{S(0)}{K} + (r - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}).$$

Remarkably enough the parameter α (or μ) does not appear in the price formula.

Our aim in this chapter is to give an elementary introduction to the Black-Scholes option theory.

5.1. Motivation of the Black-Scholes Option Prices

Suppose $g :]0, \infty[\to \mathbf{R}$ is a function such that $g(e^x) \in \mathcal{E}$. We then write $g \in \mathcal{P}$ and g is called a payoff function. In this section we want to find a natural price of a simple European contingent claim with the payoff $g(S(T))$ at time of maturity $T \in]0, \infty[$.

The present time is denoted by t and it is assumed that $0 \leq t < T$. We let $\tau = T - t$ be the residual time. The processes

$$(W(\lambda) - W(t))_{t \leq \lambda \leq T}$$

and

$$(\sqrt{\tau} W(\frac{\lambda - t}{\tau}))_{t \leq \lambda \leq T}$$

are equivalent in distribution and as

$$S(\lambda) = S(t) e^{\alpha(\lambda - t) + \sigma(W(\lambda) - W(t))}, \ t \leq \lambda \leq T$$

we may assume that

$$S(\lambda) = S(t) e^{\alpha(\lambda - t) + \sigma\sqrt{\tau} W(\frac{\lambda - t}{\tau})}, \ t \leq \lambda \leq T.$$

Next choose $N \in \mathbf{N}_+$ and define $h = \tau / N$ and

$$t_n = t + nh, \ n = 0, 1, ..., N.$$

In the following we approximate the process

$$(\sqrt{\tau} W(\frac{\lambda - t}{\tau}))_{t \leq \lambda \leq T}$$

by the process

$$(\sqrt{\tau} W_N(\frac{\lambda - t}{\tau}))_{t \leq \lambda \leq T}$$

where W_N is as in Section 4.1. Recall that $(W_N(\lambda))_{0 \leq \lambda \leq N}$ possesses continuous sample functions, which are affine in each time interval $\left[\frac{n}{N}, \frac{n+1}{N}\right]$, $n = 0, 1, 2, ..., N$, and, moreover,

$$W_N(\frac{n}{N}) = \frac{1}{\sqrt{N}} \sum_{1 \leq k \leq n} X_k$$

where $(X_k)_{k=1}^N$ is an i.i.d. such that

$$P[X_1 = 1] = P[X_1 = -1] = \frac{1}{2}.$$

In the next step we introduce an approximate stock price process by the equation

$$S_N(\lambda) = S(t)e^{\alpha(\lambda-t)+\sigma\sqrt{\tau}W_N(\frac{\lambda-t}{\tau})},\ t \le \lambda \le T.$$

Observe that the process

$$(\alpha(\lambda - t) + \sigma\sqrt{\tau}W_N(\frac{\lambda - t}{\tau}))_{t\le\lambda\le T}$$

is affine in each time interval

$$[t_n, t_{n+1}],\ n = 0, ..., N-1$$

and

$$S_N(t_n) = S(t)e^{\alpha nh+\sigma\sqrt{h}\sum_{1\le k\le n} X_k},\ n = 0, 1, ..., N.$$

In view of the binomial model it is natural to introduce a time discrete stock price process

$$\tilde{S}(n) = S_N(t_n), n = 0, ..., N$$

and a time discrete bond price process

$$\tilde{B}(n) = B(t_n),\ n = 0, ..., N$$

and we observe that

$$\tilde{S}(n+1) = \tilde{S}(n)e^{\alpha h+\sigma\sqrt{h}X_{n+1}}.$$

and

$$\tilde{B}(n+1) = \tilde{B}(n)e^{rh}.$$

Thus we have got a binomial model with N periods, where

$$u = \alpha h + \sigma\sqrt{h}$$

and

$$d = \alpha h - \sigma\sqrt{h}$$

and, assuming N large,

$$\alpha h + \sigma\sqrt{h} > rh > \alpha h - \sigma\sqrt{h}$$

which implies that the model is free of arbitrage.

From Chapter 2 we know how to price a European contingent claim paying $g(\tilde{S}(N))$ at time of maturity N. If $V(n)$ denotes the price at time and

$$V^u(n+1) = V(n+1)_{|X_{n+1}=+1}$$

and

$$V^d(n+1) = V(n+1)_{|X_{n+1}=-1}.$$

we have the recurrence equation

$$V(n) = e^{-rh}(q_u V^u(n+1) + q_d V^d(n+1))$$

with

$$q_u = \frac{e^{rh} - e^{\alpha h - \sigma\sqrt{h}}}{e^{\alpha h + \sigma\sqrt{h}} - e^{\alpha h - \sigma\sqrt{h}}} = 1 - q_d$$

The results in Chapter 2 show that $V(n)$ is a deterministic function of n and $\tilde{S}(n)$ and it is natural to write

$$V(n) = v(t + nh, \tilde{S}(n)).$$

Clearly V and v depend on N.

The next sections are rather technical. First define $s = \tilde{S}(0) = S(t)$ and set $n = 0$ in the recurrence equation above to get

$$v(t,s)e^{rh} = q_u v(t+h, se^{\alpha h + \sigma\sqrt{h}}) + q_d v(t+h, se^{\alpha h - \sigma\sqrt{h}}).$$

Moreover,

$$q_u = \frac{e^{(r-\alpha)h} - e^{-\sigma\sqrt{h}}}{e^{\sigma\sqrt{h}} - e^{-\sigma\sqrt{h}}}$$

$$= \frac{1}{2}\frac{1 + (r-\alpha)h - 1 + \sigma\sqrt{h} - \frac{1}{2}\sigma^2 h + o(h)}{\sigma\sqrt{h} + o(h)}, \; h \to 0$$

or, after a simplification,

$$q_u = \frac{1}{2}\frac{1 + (r - \alpha - \frac{\sigma^2}{2})\frac{\sqrt{h}}{\sigma} + o(\sqrt{h})}{1 + o(\sqrt{h})}$$

$$= \frac{1}{2} + (r - \alpha - \frac{\sigma^2}{2})\frac{\sqrt{h}}{2\sigma} + o(\sqrt{h}),\ h \to 0$$

(the notation $f(h) = o(g(h))$, $h \to 0$, means that $f(h)/g(h) \to 0$ as $h \to 0$). Hence

$$q_d = \frac{1}{2} - (r - \alpha - \frac{\sigma^2}{2})\frac{\sqrt{h}}{2\sigma} + o(\sqrt{h}),\ h \to 0.$$

Next we assume v extends to a smooth function and

$$v(t+h, se^{\alpha h + \sigma\sqrt{h}}) = v(t,s) + \frac{\partial v}{\partial t}(t,s)h + \frac{\partial v}{\partial s}(t,s)s(e^{\alpha h + \sigma\sqrt{h}} - 1)$$

$$+\frac{1}{2}\frac{\partial^2 v}{\partial s^2}(t,s)s^2(e^{\alpha h + \sigma\sqrt{h}} - 1)^2 + o(h),\ h \to 0$$

that is,

$$v(t+h, se^{\alpha h + \sigma\sqrt{h}}) = v(t,s) + \frac{\partial v}{\partial t}(t,s)h + \frac{\partial v}{\partial s}(t,s)s((\alpha + \frac{\sigma^2}{2})h + \sigma\sqrt{h})$$

$$+\frac{1}{2}\frac{\partial^2 v}{\partial s^2}(t,s)s^2\sigma^2 h + o(h),\ h \to 0.$$

In a similar way, we assume

$$v(t+h, se^{\alpha h - \sigma\sqrt{h}}) = v(t,s) + \frac{\partial v}{\partial t}(t,s)h + \frac{\partial v}{\partial s}(t,s)s((\alpha + \frac{\sigma^2}{2})h - \sigma\sqrt{h})$$

$$+\frac{1}{2}\frac{\partial^2 v}{\partial s^2}(t,s)s^2\sigma^2 h + o(h),\ h \to 0$$

and substitute these expressions for $v(t+h, se^{\alpha h \pm \sigma\sqrt{h}})$ into the recurrence equation

$$v(t,s)e^{rh} = q_u v(t+h, se^{\alpha h + \sigma\sqrt{h}}) + q_d v(t+h, se^{\alpha h - \sigma\sqrt{h}})$$

and use

$$e^{rh} = 1 + rh + o(h),\ h \to 0$$

to obtain

$$v(t,s)(1+rh) + o(h)$$

$$= q_u \left\{ v(t,s) + \frac{\partial v}{\partial t}(t,s)h + \frac{\partial v}{\partial s}(t,s)s((\alpha + \frac{\sigma^2}{2})h + \sigma\sqrt{h}) + \frac{1}{2}\frac{\partial^2 v}{\partial s^2}(t,s)s^2\sigma^2 h \right\}$$

$$+q_d\left\{v(t,s)+\frac{\partial v}{\partial t}(t,s)h+\frac{\partial v}{\partial s}(t,s)s((\alpha+\frac{\sigma^2}{2})h-\sigma\sqrt{h})+\frac{1}{2}\frac{\partial^2 v}{\partial s^2}(t,s)s^2\sigma^2 h\right\}$$

as $h\to 0$ and, hence,

$$v(t,s)rh+o(h)=$$

$$\frac{\partial v}{\partial t}(t,s)h+\frac{\partial v}{\partial s}(t,s)s((\alpha+\frac{\sigma^2}{2})h+(q_u-q_d)\sigma\sqrt{h})+\frac{1}{2}\frac{\partial^2 v}{\partial s^2}(t,s)s^2\sigma^2 h$$

as $h\to 0$. Since

$$(\alpha+\frac{\sigma^2}{2})h+(q_u-q_d)\sigma\sqrt{h}=(\alpha+\frac{\sigma^2}{2})h+2(r-\alpha-\frac{\sigma^2}{2})\frac{\sqrt{h}}{2\sigma}\sigma\sqrt{h}+o(h)$$

$$=rh+o(h),\ h\to 0$$

(the parameter α disappears!) we finally arrive at the so called Black-Scholes equation

$$v(t,s)r=\frac{\partial v}{\partial t}(t,s)+\frac{\partial v}{\partial s}(t,s)sr+\frac{1}{2}\frac{\partial^2 v}{\partial s^2}(t,s)s^2\sigma^2$$

or, written slightly differently,

$$\frac{\partial v}{\partial t}(t,s)+\frac{\sigma^2 s^2}{2}\frac{\partial^2 v}{\partial s^2}(t,s)+rs\frac{\partial v}{\partial s}(t,s)-rv(t,s)=0,\ 0\le t<T.$$

Our original derivative satisfies the condition

$$v(T,s)=g(s)$$

and Theorem 4.3.2 gives

$$v(t,s)=e^{-r\tau}E\left[g(se^{(r-\frac{\sigma^2}{2})\tau+\sigma\sqrt{\tau}G})\right]$$

where, as usual, $G\in N(0,1)$.

Definition 5.1.1. Suppose $g\in\mathcal{P}$ and consider a simple European-style derivative with payoff $Y=g(S(T))$ at time of maturity T . The theoretic price of the derivative at time t equals $\Pi_Y(t)=v(t,S(t))$, where

$$v(t,s)=e^{-r\tau}E\left[g(se^{(r-\frac{\sigma^2}{2})\tau+\sigma\sqrt{\tau}G})\right].$$

The theoretic price is also called the Black-Scholes price.

A portfolio with $h_S(t) = v_s'(t, S(t))$ units of the stock and

$$h_B(t) = (v(t, S(t)) - h_S(t)S(t))/B(t)$$

units of the bond at time t is called a hedging portfolio. The process h_S is often denoted by Δ and is called the delta of the option.

The very last part in Definition 5.1.1 needs a comment. To this end let us return to the motivation of the Black-Scholes price just before Definition 5.1.1. We know from Chapter 2 that to each contingent claim in the binomial model there is exactly one self-financing replicating strategy. To prove something similar in the Black-Scholes model requires measure theory and stochastic analysis, theories which are far beyond the scope of these lecture notes. Instead we argue as follows. There is a self-financing strategy $(h_{\tilde{S}}(n), h_{\tilde{B}}(n))_{n=0}^N$ which replicates the derivative with payoff $g(\tilde{S}(N))$. In particular,

$$V(0) = h_{\tilde{S}}(1)\tilde{S}(0) + h_{\tilde{B}}(1)\tilde{B}(0)$$

and

$$V(1) = h_{\tilde{S}}(1)\tilde{S}(1) + h_{\tilde{B}}(1)\tilde{B}(1).$$

Thus

$$h_{\tilde{S}}(0) = h_{\tilde{S}}(1) = \frac{1}{\tilde{S}(0)} \frac{V^u(1) - V^d(1)}{e^u - e^d}$$

and letting $s = S(t) = \tilde{S}(0)$,

$$h_{\tilde{S}}(0) = \frac{1}{s} \frac{v(t+h, se^{\alpha h + \sigma\sqrt{h}}) - v(t+h, se^{\alpha h - \sigma\sqrt{h}})}{e^{\alpha h + \sigma\sqrt{h}} - e^{\alpha h - \sigma\sqrt{h}}}.$$

Now by letting $h \to 0$,

$$\lim_{h\to 0} \frac{1}{s} \frac{v(t+h, se^{\alpha h + \sigma\sqrt{h}}) - v(t+h, se^{\alpha h - \sigma\sqrt{h}})}{e^{\alpha h + \sigma\sqrt{h}} - e^{\alpha h - \sigma\sqrt{h}}} = v_s'(t, s).$$

Thus the hedging portfolio in Definition 5.1.1 should "replicate the derivative with payoff $g(S(T))$ at time T" if such a concept had been defined.

Example 5.1.1. (a) If a European-style derivative pays the amount K at the termination date T, then

$$\Pi_K(t) = e^{-r\tau} E\left[K\right] = e^{-r\tau} K.$$

Here $h_S(t) = 0$ and $h_B(t) = \frac{K}{B(T)}$. Thus $\Delta = 0$.

(b) Consider a European-style derivative which pays the amount $S(T)$ at the termination date T. Then if $s = S(t)$,

$$\Pi_{S(T)}(t) = S(t)$$

since

$$e^{-r\tau} E\left[s e^{(r-\frac{\sigma^2}{2})\tau + \sigma\sqrt{\tau} G} \right] = s.$$

Here $h_S(t) = 1$ and $h_B(t) = 0$. Thus $\Delta = 1$.

In view of Example 5.1.1 a stock and a bond may be identified with European derivatives.

Theorem 5.1.1. *Assume $t < t_* < T$. Moreover, let $g \in \mathcal{P}$ and consider a simple European derivative with payoff $Y = g(S(T))$ at termination time T and another derivative with payoff $Z = \Pi_Y(t_*)$ at time t_*. Then*

$$\Pi_Z(t) = \Pi_Y(t)$$

Note here that the derivative paying the amount Z at time t_* is a simple European-style derivative since

$$Z = e^{-r(T-t_*)} \int_{-\infty}^{\infty} g(S(t_*) e^{(r-\frac{\sigma^2}{2})(T-t_*) + \sigma\sqrt{T-t_*}\, y}) \varphi(y) dy$$

is a deterministic function of $S(t_*)$ and, accordingly from this, the price $\Pi_Z(t)$ has a meaning. To emphasize the different determination dates of the derivatives in Theorem 5.1.1 it is natural to write

$$\Pi_Y(t) = \Pi_Y(t, T)$$

and

$$\Pi_Z(t) = \Pi_Z(t, t_*).$$

Theorem 5.1.1 exhibits the so called semi-group property of option prices in the Black-Scholes model, namely

$$\Pi_{\Pi_Y(t_*)}(t, t_*) = \Pi_Y(t, T), \ t \le t_* \le T.$$

Any portfolio $\mathcal{A}$ can be view as a derivative paying the amount $V_{\mathcal{A}}(t_*)$ at a given future point of time t_* since the market price of the the portfolio at time t_* equals $V_{\mathcal{A}}(t_*)$. From this remark the semi-group property extends to a portfolio $\mathcal{A}$ consisting of stocks, bonds, and European derivatives on the stock and we get the following equation, namely

$$\Pi_{V_{\mathcal{A}}(t_*)}(t, t_*) = V_{\mathcal{A}}(t), \ t \le t_* \le T.$$

PROOF OF THEOREM 5.1.1. We have

$$\Pi_Y(t_*) = e^{-r(T-t_*)} \int_{-\infty}^{\infty} g(S(t_*)e^{(r-\frac{\sigma^2}{2})(T-t_*)+\sigma\sqrt{T-t_*}y})\varphi(y)dy$$

and it follows that Z is a deterministic function of $S(t_*)$. Thus

$$\Pi_Z(t) = e^{-r(t_*-t)}$$

$$\times \int_{-\infty}^{\infty} \left\{ e^{-r(T-t_*)} \int_{-\infty}^{\infty} g((S(t)e^{(r-\frac{\sigma^2}{2})(t_*-t)+\sigma\sqrt{t_*-t}x} e^{(r-\frac{\sigma^2}{2})(T-t_*)+\sigma\sqrt{T-t_*}y})\varphi(y)dy \right\} \varphi(x)dx$$

$$= e^{-r(T-t)} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(S(t)e^{(r-\frac{\sigma^2}{2})(t_*-t)+\sigma\sqrt{t_*-t}x} e^{(r-\frac{\sigma^2}{2})(T-t_*)+\sigma\sqrt{T-t_*}y})\varphi(x)\varphi(y)dxdy$$

$$= e^{-r(T-t)} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(S(t)e^{(r-\frac{\sigma^2}{2})(T-t)+\sigma(\sqrt{t_*-t}x+\sqrt{T-t_*}y)}) \exp(-\frac{1}{2}(x^2+y^2))\frac{dxdy}{2\pi}$$

$$= e^{-r(T-t)} E\left[g(se^{(r-\frac{\sigma^2}{2})(T-t)+\sigma(\sqrt{t_*-t}X+\sqrt{T-t_*}Y)})\right]_{|s=S(t)}$$

where $X, Y \in N(0,1)$ are independent and we get

$$\Pi_Z(t) = e^{-r(T-t)} E\left[g(se^{(r-\frac{\sigma^2}{2})(T-t)+\sigma(\sqrt{T-t}G})\right]_{|s=S(t)} = \Pi_Y(t).$$

This proves Theorem 5.1.1.

The following result gives a weak type of dominance principle for portfolios containing the stock, the bond, and European derivatives with the stock as the underlying security.

Theorem 5.1.2. *Suppose $g_i \in \mathcal{P}$, $i = 1, ..., m$, and consider m European-style derivatives with payoffs $Y_i = g_i(S(T))$, $i = 1, ..., m$, at time of maturity T. Furthermore, let a_i, $i = 1, ..., m$, be real numbers such that*

$$\sum_{i=1}^{m} a_i g_i \geq 0.$$

Then

$$\sum_{i=1}^{m} a_i \Pi_{Y_i}(t) \geq 0.$$

If

$$\sum_{i=1}^{m} a_i g_i = 0$$

then

$$\sum_{i=1}^{m} a_i \Pi_{Y_i}(t) = 0.$$

PROOF We have

$$\sum_{i=1}^{m} a_i \Pi_{Y_i}(t) = \sum_{i=1}^{m} a_i e^{-r\tau} E\left[g_i(se^{(r-\frac{\sigma^2}{2})\tau+\sigma\sqrt{\tau}G})\right]$$

$$e^{-r\tau} E\left[\sum_{i=1}^{m} a_i g_i(se^{(r-\frac{\sigma^2}{2})\tau+\sigma\sqrt{\tau}G})\right]$$

and Theorem 5.1.2 follows at once.

Recalling the notation

$$c(t, S(T), K, T) = \Pi_{(S(T)-K)^+}(t)$$

and

$$p(t, S(t), K, T) = \Pi_{(K-S(T))^+}(t)$$

Theorems 5.1.2 implies the put-call parity relation

$$c(t, S(t), K, T) - S(T) = p(t, S(T), K, T) - Ke^{-r\tau}.$$

In fact, since

$$(S(T) - K)^+ - S(T) + K - (K - S(T))^+ = 0$$

for all values on $S(T)$,

$$\Pi_{(S(T)-K)^+}(t) - \Pi_{S(T)}(t) + \Pi_K(t) - \Pi_{(K-S(T))^+}(t) = 0.$$

Now the put-call parity relation follows from Example 5.1.1.

In a similar way we conclude that the forward price $S^T_{for}(t) = S(t)e^{r\tau}$.

Exercises

1. Suppose $g \in \mathcal{P}$ and consider a simple European option with payoff $Y = g(S(T)$ and termination date T. Prove that $\Pi_Y(t) = u(t, S^T_{for}(t))$, where
$$u(t, f) = e^{-r\tau} E\left[g(fe^{-\frac{\sigma^2}{2}\tau + \sigma\sqrt{\tau}G})\right].$$
Moreover, prove that
$$\frac{\partial u}{\partial t}(t, f) + \frac{\sigma^2 f^2}{2}\frac{\partial^2 u}{\partial f^2}(t, f) - ru(t, f) = 0,\ 0 \le t < T.$$

2. Suppose $g \in \mathcal{P}$ and consider a simple European option with payoff $Y = g(S(T)$ and termination date T. Then, by Definition 5.1.1, $\Pi_Y(t) = v(t, S(t))$, where
$$v(t, s) = e^{-r\tau} E\left[g(se^{(r-\frac{\sigma^2}{2})\tau + \sigma\sqrt{\tau}G})\right].$$
(a) Prove that $v(t, s)$ is a convex function of s for fixed t if g is a convex function.

(b) In addition, assume $g(0+) = 0$. Prove that $v(t, s) \ge g(s)$.

3. Let a be a positive real number and suppose the function $u(t,s)$ satisfies the Black-Scholes differential equation

$$u'_t + \frac{\sigma^2 s^2}{2} u''_{ss} + rsu'_s - ru = 0, \;\; 0 \le t < T, \;\; s > 0.$$

Show that the function $v(t,s) = s^{1-\frac{2r}{\sigma^2}} u(t, \frac{a}{s})$ satisfies the Black-Scholes differential equation.

5.2 The Prices of Some Simple Derivatives

Theorem 5.2.1. *If* $t < T$ and $\tau = T - t$,

$$c(t,s,K,T) = s\Phi(d_1) - Ke^{-r\tau}\Phi(d_2),$$

and

$$p(t,s,K,T) = Ke^{-r\tau}\Phi(-d_2) - s\Phi(-d_1)$$

w*here*

$$d_1 = \frac{\ln\frac{s}{K} + (r + \frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}}$$

and

$$d_2 = \frac{\ln\frac{s}{K} + (r - \frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}}.$$

PROOF Definition 5.1.1 gives

$$c(t,s,K,T) = e^{-r\tau} E\left[\max(0, se^{(r-\frac{\sigma^2}{2})\tau - \sigma\sqrt{\tau}G} - K)\right]$$

$$= e^{-r\tau} \int_{-\infty}^{\infty} \max(0, se^{(r-\frac{\sigma^2}{2})\tau - \sigma\sqrt{\tau}x} - K) e^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}}$$

$$= e^{-r\tau} \int_{x \le \frac{\ln\frac{s}{K} + (r-\frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}}} (se^{(r-\frac{\sigma^2}{2})\tau - \sigma\sqrt{\tau}x} - K) e^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}}$$

$$= e^{-r\tau} \int_{x \le d_2} se^{(r-\frac{\sigma^2}{2})\tau - \sigma\sqrt{\tau}x - \frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}} - e^{-r\tau} K\Phi(d_2)$$

$$= s\int_{x\leq d_2} e^{-\frac{1}{2}(x+\sigma\sqrt{\tau})^2}\frac{dx}{\sqrt{2\pi}} - e^{-r\tau}K\Phi(d_2)$$
$$= s\Phi(d_1) - Ke^{-r\tau}\Phi(d_2).$$

Furthermore by the call-put parity relation

$$p(t,s,K) = Ke^{-r\tau} - s + c(t,s,K) = Ke^{-r\tau} - s + s\Phi(d_1) - Ke^{-r\tau}\Phi(d_2)$$
$$= Ke^{-r\tau}(1-\Phi(d_2)) - s(1-\Phi(d_1)) = Ke^{-r\tau}\Phi(-d_2) - s\Phi(-d_1)$$

which completes the proof of Theorem 5.2.1.

Below $\varphi = \Phi'$, that is

$$\varphi(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}.$$

For a general payoff function $g \in \mathcal{P}$ the price $\Pi_{g(S(T))}(t)$ of the corresponding European contingent claim equals

$$v(t,s) = e^{-r\tau}\int_{-\infty}^{\infty} g(se^{(r-\frac{\sigma^2}{2})\tau+\sigma\sqrt{\tau}x})\varphi(x)dx$$

which is simple to compute by numerical integration. Alternatively, if $G_1, ..., G_n$ are independent observations on $G \in N(0,1)$, the Monte Carlo method gives the following approximate option price, viz.

$$\frac{e^{-r\tau}}{n}\sum_{i=1}^{n} g(se^{(r-\frac{\sigma^2}{2})\tau+\sigma\sqrt{\tau}G_i}).$$

Finally, the binomial model, which motivated the Black-Schole price above, can be used to find an approximation $v(t,S(t))$ of the price $\Pi_{g(S(T))}(t)$ as follows. First choose $N \in \mathbf{N}_+$ and define $h = \tau/N$ and

$$t_n = t + nh,\ n = 0,1,...,N.$$

Then, if $s = S(t)$,

$$v(t_N, se^{(N-2j)\sigma\sqrt{h}}) = g(se^{(N-2j)\sigma\sqrt{h}}),\ j = 0,1,...,N$$

and for every $n = N-1, N-2, ..., 1, 0$, we let

$$v(t_n, se^{(n-2j)\sigma\sqrt{h}}) = e^{-rh}(q_u v(t_{n+1}, se^{(n+1-2j)\sigma\sqrt{h}}) + q_d v(t_{n+1}, se^{(n-1-2j)\sigma\sqrt{h}}))$$

for $j = 0, 1, ..., n$, where

$$q_u = \frac{e^{rh} - e^{-\sigma\sqrt{h}}}{e^{\sigma\sqrt{h}} - e^{-\sigma\sqrt{h}}}$$

and $q_d = 1 - q_u$. Thus the drift parameter α is chosen equal to 0. The quantity $v(t_0, s)$ approximates the option price $\Pi_{g(S(T))}(t)$.

As an example consider a European call with volatility 20% when the stock price equals 40 crowns at time 0. The annual simple interest rate is 5%, that is $r = \ln 1.05$. We get the following result with $N = 5, 20$, and 50 and different strikes and residuals:

$Black - Scholes\ \ price$

$K\backslash\tau$	1/12	4/12	7/12
35	5.15	5.76	6.40
40	1.00	2.17	3.00
45	0.02	0.51	1.10

$N = 5$

$K\backslash\tau$	1/12	4/12	7/12
35	5.14	5.77	6.45
40	1.05	2.26	3.12
45	0.02	0.54	1.15

$N = 20$

$K\backslash\tau$	1/12	4/12	7/12
35	5.15	5.77	6.39
40	0.99	2.14	2.97
45	0.02	0.51	1.11

$N = 50$

$K\backslash\tau$	1/12	4/12	7/12
35	5.15	5.76	6.40
40	1.00	2.16	2.99
45	0.02	0.51	1.11

The Monte Carlo method with $n = 10^6$ simulations gave us the following results:

exact prices

$K\backslash\tau$	1/12	4/12	7/12
35	5.15	5.76	6.40
40	1.00	2.17	3.00
45	0.02	0.51	1.10

10^6 random numbers

$K\backslash\tau$	1/12	4/12	7/12
35	5.15	5.75	6.40
40	1.00	2.17	3.00
45	0.02	0.51	1.11

Since the dominance principle in Chapter 1 shows that it is not optimal to exercise an American call before maturity, we define its price as the corresponding European call price (we here assume that the underlying pays no dividends). However, in this introduction to the Black-Scholes theory we cannot, in general, define the price of an American contingent claim. Already the American put causes great troubles but the following can be proved using very advanced tools. If an American put has strike price K and termination time T, the price at time t is a function $v(t, S(t))$ of $(t, S(t))$, where $v(t,s)$ solves the Black-Scholes differential equation in a domain

$$D = \{(t,s);\, s > b(t),\, 0 \le t < T\}$$

for an appropriate increasing and convex function b such that

$$\lim_{t \to T-} b(t) = K.$$

Moreover,

$$\begin{aligned} v(t,s) &> \max(K - s, 0),\, (t,s) \in D, \\ v(t,s) &= \max(K - s, 0),\, (t,s) \in \partial D \end{aligned}$$

and

$$\frac{dv}{ds}(t, b(t)+) = -1,\, 0 \le t < T.$$

Here the function b, which is unknown from the beginning, is a part of the solution. If $S(t) < b(t)$, it is optimal to exercise the put. Like the put price, the so called critical boundary $s = b(t)$ is not given by any known analytic expression (see Myneni $[MY]$, Carr, Jarrow, and Myeni $[CJM]$, and Ekström $[EK]$).

The binomial method can be used to obtain an approximate price $v(t, S(t))$ of a simple American option with payoff $g \in \mathcal{P}$. Set $s = S(t)$ and let

$$v(t_N, se^{(N-2j)\sigma\sqrt{h}}) = g(se^{(N-2j)\sigma\sqrt{h}}),\, j = 0, 1, ..., N.$$

Furthermore, for every $n = N-1, N-2, ..., 1, 0$, we let

$$v(t_n, se^{(n-2j)\sigma\sqrt{h}})$$

$$= \max(g(se^{(n-2j)\sigma\sqrt{h}}), e^{-rh}(q_u v(t_{n+1}, se^{(n+1-2j)\sigma\sqrt{h}}) + q_d v(t_{n+1}, se^{(n-1-2j)\sigma\sqrt{h}})))$$

for $j = 0, 1, ..., n$.

As an example consider an American put with volatility 20% when the stock price equals 40 crowns at time 0. The annual simple interest rate is 5% , that is $r = \ln 1.05$. If $K = 45$ and $\tau = 4/12$, the algorithm gives the price 5.08 if $N = 25$ and 5.09 if $N = 50, 75$, 100 and 150. The corresponding European put has the price 4.78. If $K = 45$ and $\tau = 1/12$ the American put has the price 5 and a closer analysis shows that it is optimal to exercise the option.

The Black-Scholes theory applies to options on exchange rates and futures contracts as will be seen below.

First suppose we have two currencies; the domestic currency Swedish crowns and the foreign currency US dollars. Let $\xi(t)$ denote the exchange rate at time t that is, at time t the value of 1 US dollar equals $\xi(t)$ Swedish crowns. The domestic interest rate r as well as the foreign interest rate r_f are positive deterministic constants, and the corresponding bond price processes are denoted by B and B_f, respectively.

Now consider the right but not the obligation to buy one US dollar at the price K Swedish crowns at time T. The value of this contract at time T equals

$$Y = \max(0, \xi(T) - K)$$

in Swedish crowns. To price this stochastic payoff we will assume that the exchange rate process $(\xi(t))_{t\geq 0}$ is a geometric Brownian motion with volatility σ and, moreover, observe that the process

$$S(t) = B_f(t)\xi(t), \ t \geq 0$$

is the price process of a traded Swedish asset. In fact, we may exchange Swedish crowns to US dollars, buy the US bond, and when selling the US bond the cash in US dollars is exchanged to Swedish crowns. Now writing

$$Y = B_f(T)^{-1} \max(0, S(T) - B_f(T)K)$$

and assuming $t < T$, the price $\Pi_Y(t)$ at time t in Swedish crowns equals

$$\Pi_Y(t) = B_f(T)^{-1} c(t, S(t), B_f(T)K), T)$$
$$= B_f(T)^{-1} \left(B_f(t)\xi(t)\Phi(D_1) - B_f(T)Ke^{-r\tau}\Phi(D_2) \right)$$

where

$$D_1 = \frac{\ln \frac{B_f(t)\xi(t)}{B_f(T)K} + (r + \frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}}$$

$$= \frac{\ln \frac{\xi(t)}{K} + (r - r_f + \frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}}$$

and

$$D_2 = \frac{\ln \frac{\xi(t)}{K} + (r - r_f - \frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}}.$$

After a simplification we get the so-called Garman-Kohlhagen price formula

$$\Pi_Y(t) = \xi(t)e^{-r_f\tau}\Phi(D_1) - Ke^{-r\tau}\Phi(D_2).$$

The option to sell one US dollar is treated similarly. The option to buy one unit of the IBM stock at a given price in Swedish crowns is slightly more involved and will be treated later in Chapter 6.

As a second example how Theorem 5.2.1 can be used consider a call option, with exercise date T and strike price K, on an underlying forward contract on S written at time T and with delivery date T_1, where $T_1 > T$. At the exercise time T the holder of the option will obtain the amount

$$Y = \max(0, S_{for}^{T_1}(T) - K)$$

and a long position in the forward contract. Since the price of the forward contract vanishes at time T we only have to evaluate the price of an option with payoff Y at time T. Since

$$Y = e^{r(T_1-T)}\max(0, S(T) - Ke^{-r(T_1-T)})$$

we get for $t < T$,

$$\Pi_Y(t) = e^{r(T_1-T)}c(t, S(t), Ke^{-r(T_1-T)}, T)$$

$$= e^{r(T_1-T)}\left\{ S(t)\Phi(\frac{\ln \frac{S(t)}{Ke^{-r(T_1-T)}} + (r + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}) \right.$$

$$\left. -Ke^{-r(T_1-T)}e^{-r(T-t)}\Phi(\frac{\ln \frac{S(t)}{Ke^{-r(T_1-T)}} + (r - \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}) \right\}$$

$$= e^{-r\tau}\left\{ S_{for}^{T_1}(t)\Phi(\frac{\ln \frac{S_{for}^{T_1}(t)}{K} + \frac{\sigma^2}{2}\tau}{\sigma\sqrt{\tau}}) - K\Phi(\frac{\ln \frac{S_{for}^{T_1}(t)}{K} - \frac{\sigma^2}{2}\tau}{\sigma\sqrt{\tau}}) \right\}.$$

This formula is called the Black-76 formula.

Exercises

1. Consider a European call on a stock with volatility 25%, termination date 20 March, and strike price 100 crowns. The simple annual rate is $r = 5\%$. During the period 20 January-20 February the stock price increases from 97 crowns to 103 crowns. Find the increase in the option price during the same period in (a) crowns. (b) percentage. Assume each month possesses the same number of trading days.

   (Answer: (a) 1.94 crowns (b) 64.73 %)

2. Consider a simple European derivative with payoff $Y = g(S(T))$, where $g \in \mathcal{P}$. Find a function $\Theta(t, s, y)$ such that $\Pi_Y(t) = v(t, S(t))$, where

$$v(t, s) = \int_0^\infty g(y)\Theta(t, s, y)dy \text{ if } t < T.$$

3. Suppose K and L are positive constants.

   (a) **(”cash or nothing call”)** A simple European option on S pays nothing at time of maturity T if $S(T) < K$ and, in other cases, it pays the amount L. Find the price of the derivative at time t.

   (b) **(”asset or nothing call”)** A simple European option on S pays nothing at time of maturity T if $S(T) < K$ and, in other cases, it pays the amount $S(T)$. Find the price of the derivative at time t.

   (Answer: Let $s = S(t)$. (a) $Le^{-r\tau}\Phi(d_2)$ (b) $s\Phi(d_1)$)

4. **(”as you like it option”** or **”chooser option”)** Let $t < T < T_1$ and $K > 0$. At time T a financial derivative gives its owner the right to choose either a European call on S with strike price K and termination date T_1 or a European put on S with strike price K and termination date T_1. Find the price of the derivative at time t.

5. A European derivative pays $Y = \cos^2(\ln S(T))$ at the termination date T. Find the price of the derivative at time t.

   (Answer: $\frac{1}{2}e^{-r\tau}\left\{1 + e^{-2\sigma^2\tau}\cos(2\ln s + (2r - \sigma^2)\tau)\right\}$)

6. Suppose $X \in N(\alpha, \sigma^2)$, where $\sigma > 0$, and set $Y = e^X$. Prove that

$$E\left[\min(K,Y)\right]$$
$$= E\left[Y\right]\Phi(\frac{\ln\frac{K}{E[Y]} - \frac{\sigma^2}{2}}{\sigma}) + K\Phi(\frac{\ln\frac{E[Y]}{K} - \frac{\sigma^2}{2}}{\sigma})$$

if $K > 0$.

5.3 The Greeks

The Greeks of an option measure the sensitivity of the option price from its parameters. Expressed in mathematical terms, if a simple European option on S with payoff function $g \in \mathcal{P}$ and termination date T has the price $v(t, S(t))$ at time t, where

$$v(t,s) = e^{-r\tau}E\left[g(se^{(r-\frac{\sigma^2}{2})\tau+\sigma\sqrt{\tau}G})\right].$$

then partial derivatives of $v = v(t, s, r, \sigma)$ evaluated at the point $(t, S(t))$, where $t < T$, are called Greeks of the option. We already have defined the delta in Definition 5.1.1 and know that this Greek is an important tool to hedge the option. The following list is (more or less) standard:

delta: $\frac{\partial v}{\partial s}$
gamma: $\frac{\partial^2 v}{\partial s^2}$
rho: $\frac{\partial v}{\partial r}$
theta: $\frac{\partial v}{\partial t}$
vega: $\frac{\partial v}{\partial \sigma}$

Furthermore the quantity $\frac{s\frac{\partial v}{\partial s}}{v}$ is called the omega of the stock price. Thus
omega: $\frac{s\frac{\partial v}{\partial s}}{v} = \text{(formally)} = \frac{\frac{\partial v}{v}}{\frac{\partial s}{s}}$.

Omega is also called the elasticity of the option price with respect to the stock price; if the stock price changes 1% the option price changes about omega %.

Since a linear combination of payoff functions is a payoff function a portfolio containing the stock, the bond, and European options on S has Greeks as above. Such a portfolio is said to be delta neutral if the delta of the portfolio vanishes.

Theorem 5.3.1. *For a European call price and* $\tau > 0$*,*

$$\begin{aligned} \text{delta} &= \Phi(d_1) \\ \text{gamma} &= \frac{\varphi(d_1)}{s\sigma\sqrt{\tau}} \\ \text{rho} &= K\tau e^{-r\tau}\Phi(d_2) \\ \text{theta} &= -\frac{s\varphi(d_1)\sigma}{2\sqrt{\tau}} - rKe^{-r\tau}\Phi(d_2) \\ \text{vega} &= s\varphi(d_1)\sqrt{\tau} \end{aligned}$$

where all expressions to the right are evaluated at the point $(t, s) = (t, S(t))$*. In particular, the delta, gamma, rho, and vega of a European call are strictly positive but theta is strictly negative.*

The most common Greeks for a European put are obtained from the put-call parity relation

$$p(t, s, K, T) = c(t, s, K, T) - s + Ke^{-r\tau}$$

and Theorem 5.3.1.

PROOF OF THEOREM 5.3.1 If c denotes the call price,

$$c = s\Phi(d_1) - Ke^{-r\tau}\Phi(d_2)$$

where

$$d_1 = \frac{\ln\frac{s}{K} + (r + \frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}}$$

and

$$d_2 = \frac{\ln\frac{s}{K} + (r - \frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}} = d_1 - \sigma\sqrt{\tau}.$$

Hence

$$\frac{\partial c}{\partial s} = \Phi(d_1) + s\varphi(d_1)\frac{\partial d_1}{\partial s} - Ke^{-r\tau}\varphi(d_2)\frac{\partial d_2}{\partial s}$$

$$= \Phi(d_1) + \frac{1}{\sqrt{2\pi}}\frac{\partial d_1}{\partial s}\left\{se^{-\frac{d_1^2}{2}} - Ke^{-r\tau}e^{-\frac{d_2^2}{2}}\right\}$$

$$= \Phi(d_1) + \frac{1}{\sqrt{2\pi}}\frac{\partial d_1}{\partial s}\left\{se^{-\frac{d_1^2}{2}} - Ke^{-r\tau}e^{-\frac{d_1^2}{2}+d_1\sigma\sqrt{\tau}-\frac{\sigma^2\tau}{2}}\right\}$$

$$= \Phi(d_1) + \frac{e^{-\frac{d_1^2}{2}}}{\sqrt{2\pi}}\frac{\partial d_1}{\partial s}\left\{s - Ke^{-r\tau+d_1\sigma\sqrt{\tau}-\frac{\sigma^2\tau}{2}}\right\}$$

$$= \Phi(d_1) + \frac{e^{-\frac{d_1^2}{2}}}{\sqrt{2\pi}}\frac{\partial d_1}{\partial s}\left\{s - Ke^{-r\tau+\ln\frac{s}{K}+(r+\frac{\sigma^2}{2})\tau-\frac{\sigma^2\tau}{2}}\right\} = \Phi(d_1).$$

This proves the formula for the delta. The formula for the gamma now follows from

$$\frac{\partial d_1}{\partial s} = \frac{1}{s\sigma\sqrt{\tau}}.$$

The remaining part of Theorem 5.3.1 is left as an exercise.

For a general simple European-style derivative the Greeks of the option price are obtained by numerical integration and differentiation. Alternatively the following result may be used.

Theorem 5.3.2. *A simple European option on S with payoff function $g \in \mathcal{P}$ and termination date T has the price $v(t, S(t))$ at time t, where*

$$v(t, S(t)) = e^{-r\tau}E\left[g(se^{(r-\frac{\sigma^2}{2})\tau+\sigma\sqrt{\tau}G})\right]_{|s=S(t)}.$$

Moreover, if $\tau > 0$.

$$\text{delta} = \frac{e^{-r\tau}}{S(t)\sigma\sqrt{\tau}}E\left[g(se^{(r-\frac{\sigma^2}{2})\tau+\sigma\sqrt{\tau}G})G\right]_{|s=S(t)}$$

$$\text{gamma} = \frac{e^{-r\tau}}{S^2(t)\sigma\tau}E\left[g(se^{(r-\frac{\sigma^2}{2})\tau+\sigma\sqrt{\tau}G})(\frac{\tau G^2}{\sigma\tau} - \sqrt{\tau}G - \frac{1}{\sigma})\right]_{|s=S(t)}$$

$$\text{rho} = e^{-r\tau} E\left[g(se^{(r-\frac{\sigma^2}{2})\tau+\sigma\sqrt{\tau}G})(\frac{\sqrt{\tau}G}{\sigma} - \tau)\right]_{|s=S(t)}$$

$$\text{theta} = \left\{ r + \frac{1}{2\tau}\right\} v(t,S(t)) - e^{-r\tau} E\left[g(se^{(r-\frac{\sigma^2}{2})\tau+\sigma\sqrt{\tau}G}) \left\{ \frac{G^2}{2\tau} + \frac{(r-\frac{\sigma^2}{2})G}{\sigma\sqrt{\tau}}\right\}\right]_{|s=S(t)}.$$

and

$$\text{vega} = e^{-r\tau} E\left[g(se^{(r-\frac{\sigma^2}{2})\tau+\sigma W(\tau)})(\frac{\tau G^2}{\sigma\tau} - \sqrt{\tau}G - \frac{1}{\sigma})\right]_{|s=S(t)}$$

PROOF We here compute only the theta. The other Greeks in Theorem 5.3.2 are obtained in a similar way. First

$$v(t,s) = e^{-r\tau}\int_{-\infty}^{\infty} g(e^{\ln s + (r-\frac{\sigma^2}{2})\tau + \sigma\sqrt{\tau}x})\varphi(x)dx$$

where

$$\varphi(x) = \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}.$$

We now set $\ln y = \ln s + (r - \frac{\sigma^2}{2})\tau + \sigma\sqrt{\tau}x$ and have

$$v(t,s) = \frac{e^{-r\tau}}{\sigma\sqrt{\tau}}\int_0^{\infty} g(y)\varphi(\frac{\ln\frac{y}{s} - (r-\frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}})\frac{dy}{y}.$$

Thus

$$\frac{\partial v}{\partial t}(t,s) = \left\{ \frac{re^{-r\tau}}{\sigma\sqrt{\tau}} + \frac{e^{-r\tau}}{2\sigma\sqrt{\tau}\tau}\right\}\int_0^{\infty} g(y)\varphi(\frac{\ln\frac{y}{s} - (r-\frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}})\frac{dy}{y}$$

$$-\frac{e^{-r\tau}}{\sigma\sqrt{\tau}}\int_0^{\infty} g(y)\frac{\ln\frac{y}{s} - (r-\frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}}\varphi(\frac{\ln\frac{y}{s} - (r-\frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}})\left\{ \frac{\ln\frac{y}{s} - (r-\frac{\sigma^2}{2})\tau}{2\sigma\sqrt{\tau}\tau} + \frac{r-\frac{\sigma^2}{2}}{\sigma\sqrt{\tau}}\right\}\frac{dy}{y}$$

$$= \left\{ r + \frac{1}{2\tau}\right\}\frac{e^{-r\tau}}{\sigma\sqrt{\tau}}\int_0^{\infty} g(y)\varphi(\frac{\ln\frac{y}{s} - (r-\frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}})\frac{dy}{y}$$

$$-\frac{e^{-r\tau}}{\sigma\sqrt{\tau}}\int_0^{\infty} g(y)\frac{\ln\frac{y}{s} - (r-\frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}}\varphi(\frac{\ln\frac{y}{s} - (r-\frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}})\left\{ \frac{\ln\frac{y}{s} - (r-\frac{\sigma^2}{2})\tau}{2\sigma\sqrt{\tau}\tau} + \frac{r-\frac{\sigma^2}{2}}{\sigma\sqrt{\tau}}\right\}\frac{dy}{y}$$

$$= \left\{ r + \frac{1}{2\tau}\right\} v(t,s)$$

$$-e^{-r\tau}\int_0^\infty g(se^{(r-\frac{\sigma^2}{2})\tau+\sigma\sqrt{\tau}x})x\varphi(x)\left\{\frac{x}{2\tau}+\frac{r-\frac{\sigma^2}{2}}{\sigma\sqrt{\tau}}\right\}dx =$$

$$\left\{r+\frac{1}{2\tau}\right\}v(t,s)$$

$$-e^{-r\tau}E\left[g(se^{(r-\frac{\sigma^2}{2})\tau+\sigma\sqrt{\tau}G})\left\{\frac{G^2}{2\tau}+\frac{(r-\frac{\sigma^2}{2})G}{\sigma\sqrt{\tau}}\right\}\right].$$

Example 5.3.1. Suppose $0 < t < T$ and consider a financial European-style derivative with payoff $Y = (S(T) - S(0))^2/S(T)$ at time of maturity T. We want to find the price $\Pi_Y(t)$ and the delta $\Delta(t)$ of the derivative at time t.

To solve these problems first note that

$$Y = S(T) - 2S(0) + S(0)^2 S(T)^{-1}.$$

Here

$$\Pi_{S(T)}(t) = S(t)$$

and

$$\Pi_{S(0)}(t) = S(0)e^{-r\tau}$$

where $\tau = T - t$. Moreover,

$$\Pi_{S(T)^{-1}}(t) = e^{-r\tau}\int_{-\infty}^{\infty}(S(t)e^{(r-\frac{\sigma^2}{2})\tau+\sigma\sqrt{\tau}x})^{-1}e^{-\frac{x^2}{2}}\frac{dx}{\sqrt{2\pi}}$$

$$= S(t)^{-1}e^{-r\tau}e^{-(r-\frac{\sigma^2}{2})\tau}\int_{-\infty}^{\infty}e^{-\frac{x^2}{2}-\sigma\sqrt{\tau}x}\frac{dx}{\sqrt{2\pi}}$$

$$= S(t)^{-1}e^{(\sigma^2-2r)\tau}\int_{-\infty}^{\infty}e^{-\frac{1}{2}(x+\sigma\sqrt{\tau})^2}\frac{dx}{\sqrt{2\pi}} = S(t)^{-1}e^{(\sigma^2-2r)\tau}.$$

Hence

$$\Pi_Y(t) = S(t) - 2S(0)e^{-r\tau} + S(0)^2 e^{(\sigma^2-2r)\tau}S(t)^{-1}.$$

Now, if $\Pi_Y(t) = v(t, S(t))$,

$$\Delta(t) = \frac{\partial v}{\partial s}(t, S(t)) = 1 - S(0)^2 e^{(\sigma^2-2r)\tau}S(t)^{-2}.$$

Example 5.3.2. A stock price process $(S(t))_{t\geq 0}$ is governed by the equation

$$S(t) = S(0)e^{(\mu - \frac{\sigma^2}{2})t + \sigma W(t)},\ t \geq 0,$$

where $\mu > r$. If T and K denote strictly positive real numbers, we want to show that

$$E\left[(S(T) - K)^+\right] > e^{rT}c(0, S(0), K, T).$$

To this end, if $a > 0$ and

$$f(x, a) = (S(0)e^{(a - \frac{\sigma^2}{2})T - \sigma\sqrt{T}x} - K)^+,\ x \in \mathbf{R},$$

then

$$E\left[(S(T) - K)^+\right] = \int_{-\infty}^{\infty} f(x, \mu)\varphi(x)dx$$

and

$$e^{rT}c(0, S(0), K, T) = \int_{-\infty}^{\infty} f(x, r)\varphi(x)dx.$$

Since $\mu > r$ we have $f(x, \mu) \geq f(x, r)$ with strict inequality if

$$x < \frac{1}{\sigma\sqrt{T}}(\ln\frac{S(0)}{K} + (\mu - \frac{\sigma^2}{2})T).$$

Hence

$$\int_{-\infty}^{\infty} f(x, \mu)\varphi(x)dx > \int_{-\infty}^{\infty} f(x, r)\varphi(x)dx$$

which proves that

$$E\left[(S(T) - K)^+\right] > e^{rT}c(0, S(0), K, T).$$

An alternative solution based on the results in the present section is as follows. For any strictly positive real number a the Black-Scholes theory yields

$$f(a) =_{def} e^{-aT}\int_{-\infty}^{\infty}(S(0)e^{(a - \frac{\sigma^2}{2})T - \sigma\sqrt{T}x} - K)^+dx$$

$$= S(0)\Phi(d_1(a)) - Ke^{-aT}\Phi(d_2(a))$$

where

$$d_1(a) = \frac{\ln\frac{S(0)}{K} + (a + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}$$

and

$$d_2(a) = \frac{\ln \frac{S(0)}{K} + (a - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}} = d_1(a) - \sigma\sqrt{T}.$$

Hence

$$f'(a) = S(0)\varphi(d_1(a))\frac{1}{\sigma\sqrt{T}} - Ke^{-aT}\varphi(d_2(a))\frac{1}{\sigma\sqrt{T}} + KTe^{-aT}\Phi(d_2(a)).$$

But

$$S(0)\varphi(d_1(a)) - Ke^{-aT}\varphi(d_2(a)) = 0$$

(cf the proof of Theorem 5.3.1 in the textbook) and we get

$$f'(a) = KTe^{-aT}\Phi(d_2(a)).$$

Hence

$$\frac{d}{da}(e^{aT}f(a)) = Te^{aT}f(a) + KT\Phi(d_2(a)) > 0$$

and if $\mu > r$, we get

$$E\left[(S(T) - K)^+\right] = e^{\mu T}f(\mu) > e^{rT}f(r) = e^{rT}c(0, S(0), K, T).$$

Exercises

1. Show that
$$\frac{\partial c}{\partial K} = -e^{-r\tau}\Phi(d_2).$$
Use this to prove that the map $K \to c(t, S(t), K, T)$, $K > 0$, is convex in the Black-Scholes model (cf Chapter 1).

2. Let $a, K, T > 0$ be given numbers and consider a simple European-style derivative with time of maturity T and payoff K if $S(T) < a$ and payoff 0 if $S(T) \geq a$. (a) Find the price of the derivative at time $t < T$. (b) Find the delta of the derivative at time $t < T$. (c) Find the vega of the derivative at time $t < T$.

3. A European option on S pays the amount 1 at maturity T if $S(T) \leq K$ and, otherwise, it pays nothing. Suppose $t < T$. For which value on $S(t)$ is the delta of the option minimal?

   (Answer: $S(t) = Ke^{-(r+\frac{\sigma^2}{2})\tau}$)

4. Let $a, K, T > 0$. A financial European-style derivative pays the amount $Y = (\min(S(T) - K, a))^+$ at time of maturity T. Show that the delta of the derivative is positive and does not exceed

$$\frac{\ln(1 + \frac{a}{K})}{\sigma\sqrt{2\pi(T-t)}}$$

at time $t < T$.

5. Show that
$$\frac{c(t, s, K, T)}{c(t, s_0, K, T)} > \frac{s}{s_0} \text{ if } s > s_0 \text{ and } t < T.$$

6. A function $f :]a, b[\to]0, \infty[$ is said to be log-convex if $\ln f$ is convex. (a) Show that the function $f :]a, b[\to]0, \infty[$ is log-convex if and only if the function $f(x) \exp(cx)$, $a < x < b$, is convex for all real c. (b) Show that the sum of two log-convex functions is convex.

7. A European simple derivative on S has the payoff $g(S(T))$ at time of maturity T, where $g \in \mathcal{P}$ and $g(y) > 0$ for all $y > 0$. Prove that the omega is an increasing function of the stock price $S(t)$ if $\ln g(e^x)$ is a convex function of x.

5.4 Path dependent options

A subset of $\mathbf{R}^n$ of the type

$$I = \{x;\ x = (x_1, ..., x_n) \text{ and } a_k < x_k < b_k,\ k = 1, .., n\}$$

where $-\infty \le a_k < b_k \le \infty$, $k = 1, .., n$, is called an open n-cell.

Let $g(s_1, ..., s_n)$, $s_1 > 0, ...s_n > 0$, be a function satisfying the following conditions:

(a) There are finitely many mutually disjoint n-cells $I_1, ..., I_m$ such that the union of the closure of the $I_k : s$ equals $\mathbf{R}^n$.

(b) The restriction of g to each I_k is the restriction to I_k of a continuous function in the closure of I_k for $k = 1, ..., m$.

(c) There exists a positive constant A such that

$$\sup\left\{e^{-A(|x_1|+...+|x_n|)} \mid g(e^{x_1},...,e^{x_n}) \mid;\ x_1,...,x_n \in \mathbf{R}\right\} < \infty.$$

If these conditions are fulfilled we write $g \in \mathcal{P}_n$ and g is called an n-payoff function.

Suppose $t = t_0 < t_1 < ... < t_{n-1} < t_n = T$ and $\tau = T - t$. In this section we will consider a European contingent claim with payoff $Y = g(S(t_1),...,S(t_n))$ at termination time T, where g is an n-payoff function. We define the derivative price $\Pi_Y(t) = \Pi_Y(t,T)$ to be equal to the price of a European contingent claim with the payoff $\Pi_Y(t_{n-1},T)$ at time t_{n-1}, that is $\Pi_Y(t) = \Pi_{\Pi_Y(t_{n-1},T)}(t,t_{n-1})$. By induction we find that $\Pi_Y(t) = v(t,S(t))$ where

$$v(t,s) = e^{-r\tau}E\left[g((se^{(r-\frac{\sigma^2}{2})(t_k-t)+\sigma(W(t_k)-W(t))})_{k=1}^n)\right]$$

and

$$(se^{(r-\frac{\sigma^2}{2})(t_k-t)+\sigma(W(t_k)-W(t))})_{k=1}^n$$

$$= (se^{(r-\frac{\sigma^2}{2})(t_1-t)+\sigma(W(t_1)-W(t))}, se^{(r-\frac{\sigma^2}{2})(t_2-t)+\sigma(W(t_2)-W(t))}, ..., se^{(r-\frac{\sigma^2}{2})(t_n-t)+\sigma(W(t_n)-W(t))}).$$

Moreover, introducing

$$\begin{aligned} f(s,x_1,...,x_n) &= g(se^{(r-\frac{\sigma^2}{2})(t_1-t)+\sigma\sqrt{t_1-t}x_1}, se^{(r-\frac{\sigma^2}{2})(t_2-t)+\sigma\sqrt{t_1-t}x_1+\sigma\sqrt{t_2-t_1}x_2}, \\ &\quad ..., se^{(r-\frac{\sigma^2}{2})(t_n-t)+\sigma\sqrt{t_1-t}x_1+...+\sigma\sqrt{t_n-t_{n-1}}x_n}). \end{aligned}$$

we also have

$$v(t,s)$$

$$= e^{-r\tau}\int_{\mathbf{R}}...\int_{\mathbf{R}} f(s,x_1,...,x_n)\exp(-\frac{1}{2}(x_1^2+...+x_n^2))\frac{dx_1...dx_n}{\sqrt{2\pi}^n}.$$

For $n = 1$ the definition above coincides with the old one.

Example 5.4.1. A European-style derivative pays the amount $Y = \frac{S(T)}{S(T/2)}$ at time of maturity T. To find $\Pi_Y(0)$ note that for any $t \in [0,T]$ and real number a, $\Pi_{aS(T)}(t) = aS(t)$ and, hence,

$$\Pi_Y(T/2) = \Pi_{\frac{1}{S(T/2)}S(T)}(T/2) = \frac{1}{S(T/2)}\Pi_{S(T)}(T/2)$$

$$= \frac{1}{S(T/2)} S(T/2) = 1.$$

Accordingly from this,

$$\Pi_Y(0) = e^{-\frac{rT}{2}}.$$

Alernatively, using the general theory above,

$$\Pi_Y(0) = e^{-rT} E\left[g((se^{(r-\frac{\sigma^2}{2})t_k + \sigma W(t_k)})_{k=1}^2)\right]$$

with $t_1 = \frac{T}{2}$, $t_2 = T$, and $g(x_1, x_2) = \frac{x_2}{x_1}$. Hence

$$\begin{aligned}\Pi_Y(0) &= e^{-rT} E\left[\frac{se^{(r-\frac{\sigma^2}{2})T + \sigma W(T)}}{se^{(r-\frac{\sigma^2}{2})\frac{T}{2} + \sigma W(\frac{T}{2})}}\right] \\ &= e^{-r\frac{T}{2} - \frac{\sigma^2}{2}\frac{T}{2}} E\left[e^{\sigma(W(T) - W(\frac{T}{2}))}\right] \\ &= e^{-r\frac{T}{2} - \frac{\sigma^2}{2}\frac{T}{2}} E\left[e^{\sigma W(\frac{T}{2})}\right] = e^{-\frac{rT}{2}}.\end{aligned}$$

Example 5.4.2. Suppose $t < t_* < T$ and consider a European contingent claim with payoff $Y = \max(S(t_*), S(T))$ at time T. We have

$$Y = S(t_*) + \max(0, S(T) - S(t_*))$$

and

$$\begin{aligned}\Pi_Y(t_*) &= S(t_*)e^{-r(T-t_*)} + c(t_*, S(t_*), S(t_*), T) \\ &= S(t_*)(e^{-r(T-t_*)} + c(t_*, 1, 1, T)).\end{aligned}$$

Thus by Theorem 5.1.2

$$\Pi_Y(t) = S(t)(e^{-r(T-t_*)} + c(t_*, 1, 1, T)).$$

Moreover,

$$c(t_*, 1, 1, T)) = \Phi((\frac{r}{\sigma} + \frac{1}{2}\sigma)\sqrt{T - t_*}) - e^{-r(T-t_*)}\Phi((\frac{r}{\sigma} - \frac{1}{2}\sigma)\sqrt{T - t_*})$$

and it follows that

$$\Pi_Y(t) = S(t)(\Phi((\frac{r}{\sigma} + \frac{1}{2}\sigma)\sqrt{T - t_*}) + e^{-r(T-t_*)}(1 - \Phi((\frac{r}{\sigma} - \frac{1}{2}\sigma)\sqrt{T - t_*})))$$
$$= S(t)(\Phi((\frac{r}{\sigma} + \frac{1}{2}\sigma)\sqrt{T - t_*}) + e^{-r(T-t_*)}\Phi((-\frac{r}{\sigma} + \frac{1}{2}\sigma)\sqrt{T - t_*})).$$

Example 5.4.3. Consider a European-style derivative with the payoff

$$Y = \frac{1}{n}\sum_{k=1}^{n} S(\frac{kT}{n})$$

at time of maturity T. To find $\Pi_Y(0)$ we introduce a derivative paying the amount $Y_k = S(\frac{kT}{n})$ at time T. Then

$$\Pi_Y(0) = \frac{1}{n}\sum_{k=1}^{n} \Pi_{Y_k}(0).$$

Moreover, $\Pi_{Y_k}(\frac{kT}{n}) = e^{-(T-\frac{kT}{n})r} S(\frac{kT}{n})$ and, hence,

$$\Pi_{Y_k}(0) = e^{-(T-\frac{kT}{n})r} S(0).$$

Thus

$$\Pi_Y(0) = \frac{S(0)}{n}\sum_{k=1}^{n} e^{-(1-\frac{k}{n})Tr}$$
$$= \frac{S(0)}{n}\sum_{i=0}^{n-1} e^{-iTr/n} = \frac{S(0)}{n}\frac{1 - e^{-Tr}}{1 - e^{-Tr/n}}.$$

Example 5.4.4. Consider a European-style derivative type with the payoff

$$Z = \left\{\prod_{k=1}^{n} S(\frac{kT}{n})\right\}^{\frac{1}{n}}$$

at time of maturity T. To find find $\Pi_Z(0)$ let $S(0) = s$ so that

$$\Pi_Z(0) = e^{-rT} E\left[\left\{\prod_{k=1}^{n} s e^{(r-\frac{\sigma^2}{2})\frac{kT}{n} + \sigma W(\frac{kT}{n})}\right\}^{\frac{1}{n}}\right]$$

$$= se^{-rT+(r-\frac{\sigma^2}{2})\frac{(n+1)T}{2n}} E\left[e^{\frac{\sigma}{n}\sum_{k=1}^{n} W(\frac{kT}{n})}\right].$$

Set $V_i = W(\frac{iT}{n})$, $i = 0, ..., n$. Then

$$\sum_{k=1}^{n} W(\frac{kT}{n}) = V_1 + ... + V_n$$

$$= V_1 + ... + V_{n-2} + 2V_{n-1} + (V_n - V_{n-1})$$

$$= V_1 + ... + V_{n-3} + 3V_{n-2} + 2(V_{n-1} - V_{n-2}) + (V_n - V_{n-1})$$

$$= n(V_1 - V_0) + ... + 2(V_{n-1} - V_{n-2}) + (V_n - V_{n-1})$$

and using the formula $1^2 + 2^2 + ... + n^2 = \frac{n(n+1)(2n+1)}{6}$ we get

$$E\left[e^{\frac{\sigma}{n}\sum_{k=1}^{n} W(\frac{kT}{n})}\right] = \prod_{k=1}^{n} E\left[e^{\frac{\sigma(n+1-k)}{n}(V_k - V_{k-1})}\right] = e^{\frac{\sigma^2}{2n^2}(n^2+...+2^2+1^2)\frac{T}{n}}$$

$$= e^{\frac{\sigma^2}{2n^2}\frac{n(n+1)(2n+1)}{6}\frac{T}{n}} = e^{\sigma^2 T\frac{(n+1)(2n+1)}{12n^2}}.$$

Thus

$$\Pi_Z(0) = se^{-rT+(r-\frac{\sigma^2}{2})\frac{(n+1)T}{2n}+\sigma^2 T\frac{(n+1)(2n+1)}{12n^2}} = S(0)e^{(\frac{1-n}{2n}r+\frac{1-n^2}{12n^2}\sigma^2)T}.$$

Exercises

1. Set $g(\tau) = c(t, 1, 1; T)$ if $T \geq \tau = T - t \geq 0$ and $\sigma > 0$. (a) Prove that $c(t, s, s; T) = sg(T - t)$, where σ is the same in the right-hand and left-hand side. (b) Let $t \leq t_* < T$. A European derivative pays the amount $Y = \max(0, S(T) - S(t_*))$ at time T. Prove that $\Pi_Y(t) = S(t)g(T - t_*)$.(c) **(”tandem option”)** Let $\tau = T - t > 0$ and set $t_j = t + \frac{j}{n}\tau$, $j = 0, ..., n$, where $n \in \mathbf{N}_+$. A derivative pays $\max(0, S(t_j) - S(t_{j-1}))$ at time t_j for $j = 1, ..., n$. Show that the price of the derivative at time t equals $nS(t)g(\tau/n)$.

2. Suppose $t_0 < t_* < T$ and consider a financial European-style derivative with payoff $Y = \mid S(T) - S(t_*) \mid$ at time of maturity T. Find the delta $\Delta(t)$ of the derivative at time t if
   (a) $t \in]t_*, T[$.
   (b) $t \in]t_0, t_*[$.
   (c) Finally, compute $\Delta(t_*-) - \Delta(t_*+)$.

3. (Black-Scholes model) Suppose $0 < T_0 < T$ and consider a simple European-style derivative with the payoff $Y = \min(S(T_0), S(T))$ at time of maturity T. Find $\Pi_Y(t)$ for all $t \in [0, T_0]$.
   (Answer: $\Pi_Y(t) = (1 - c(T_0, 1, 1, T))S(t)$ if $0 \leq t \leq T_0$, where $c(T_0, 1, 1, T) = \Phi(\frac{r + \frac{\sigma^2}{2}}{\sigma}\sqrt{T - T_0}) - e^{-r(T-T_0)}\Phi(\frac{r - \frac{\sigma^2}{2}}{\sigma}\sqrt{T - T_0})$.)

4. (Black-Scholes model) Suppose $K > 0$ and $0 = t_0 < t_1 < ... < t_n = T$. A financial European-style derivative pays the amount Y at time of maturity T, where
$$Y = \sum_{i=1}^{n} (S(t_i) - KS(t_{i-1}))^+.$$
   Find $\Pi_Y(0)$.

5.5 Implied volatility

Consider a call on S with strike K and time of maturity T in the Black-Scholes model. If the residual time τ is strictly positive,
$$\frac{\partial c}{\partial \sigma} = s\varphi(d_1)\sqrt{\tau} = se^{-d_1^2/2}\sqrt{\frac{\tau}{2\pi}} > 0$$
and, moreover,
$$\lim_{\sigma \to 0+} c(t, s, K, T) = \lim_{\sigma \to 0+} e^{-r\tau} E\left[(se^{(r - \frac{\sigma^2}{2})\tau + \sigma\sqrt{\tau}G} - K)^+\right] = \max(0, s - Ke^{-r\tau})$$

and

$$\lim_{\sigma\to\infty} c(t,s,K,T) = \lim_{\sigma\to\infty} \left\{ s\Phi(d_1) - Ke^{-r\tau}\Phi(d_2) \right\} = s.$$

The parameter σ that makes our theoretical call price agree with the market price is called implied volatility and is denoted by σ_{imp}. If the Black-Scholes model gives a perfect description of real option markets the implied volatility would be more or less independent of the residual time τ and the strike price K. However, statistical investigations show that the implied volatility often deviates significantly from a constant function of (τ, K) and, in addition, behaves like a random function with non-negligible variance. This makes it plausible to model the volatility as a stochastic process.

If we first assume the log-price $X(t) = \ln S(t) = \alpha t + \sigma W(t)$ is a Brownian motion with drift as usual, then

$$X(t+\Delta t) - X(t) = \alpha \Delta t + \sigma(W(t+\Delta t) - W(t))$$

and it is tempting to write

$$dX(t) = \alpha dt + \sigma dW(t)$$

and

$$X(t) - X(0) = \int_0^t \alpha du + \int_0^t \sigma dW(u).$$

Here it is simple to understand the first integral

$$\int_0^t \alpha du$$

if α is replaced by a stochastic process $(\alpha(u))_{u\geq 0}$. However, in a context like here, given t the path

$$\alpha(u),\ 0 \leq u \leq t$$

only depends on the known information $\mathcal{F}_t$ at time t. We then say that the drift process $(\alpha(u))_{u\geq 0}$ is non-anticipating. Generally, the definition of information in mathematics requires measure theory and falls beyond the scope of these lecture notes. Here let us only remark that the drift process is non-anticipating, if for each t, $\alpha(t)$ is a deterministic function of the historical stock price process $(S(u))_{0\leq u\leq t}$.

If $0 = t_0 < t_1 < ... < t_n = t$ and $(\sigma(u))_{u \geq 0}$ is a non-anticipating stochastic process, which is constant in each subinterval $[t_{k-1}, t_k[$, $k = 1, ..., n$, we define

$$\int_0^t \sigma(u)dW(u) = \sum_{k=1}^{n} \sigma(t_k)(W(t_{k+1}) - W(t_k)).$$

The stochastic integral

$$\int_0^t \sigma(u)dW(u)$$

may be defined for much more general non-anticipating integrands but we have not the appropriate mathematical machinery to go any further here.

Statistical investigations support that the probability density of a log-price increment $X(t + \Delta t) - X(t)$ has more mass close to an estimated expectation of the increment than in the Black-Scholes model, when the increment is Gaussian. In fact, this phenomenon is very natural since there are transaction costs, taxes, and other frictions on the real market.

There is a variety of different stochastic volatility models. So called local volatility models and many other models are discussed in the Gatherhal book "The Volatility Surface" (see References; Books in Mathematical Finance). In the Hobson-Rogers model $[HR]$ $\sigma(t)$ is small if the stock price changes have been small the nearest period back in time and it is larger in the opposite case. At present, the Heston model $[H]$ seems to be one of the most popular stochastic volatility models on the option markets. In Heston's model the volatility process depends on more random sources than the stock price process.

Exercises

1. Assume the Black-Scholes model and show that

$$\frac{\partial^2 c}{\partial \sigma^2} = \frac{sd_1d_2}{\sigma}\varphi(d_1)\sqrt{\tau}.$$

Conclude that the map $\sigma \to c(t, s, K, T)$ is convex in the interval $]0, \sigma_0]$ and concave in the interval $[\sigma_0, \infty[$, where

$$\sigma_0 = \sqrt{\frac{2}{\tau} \mid \ln \frac{se^{r\tau}}{K} \mid}.$$

5.6 Problems with solutions

Throughout in this section the Black-Scholes model is assumed.

1. A European-style derivative pays the amount

$$Y = S(T) + \frac{1}{S(T)}$$

at time of maturity T. Find $\Pi_Y(t)$ for all $0 \leq t < T$.

Solution. We have

$$\Pi_Y(t) = \Pi_{S(T)}(t) + \Pi_{\frac{1}{S(T)}}(t).$$

Here, if $\tau = T - t$, $s = S(t)$, and $G \in N(0,1)$,

$$\Pi_{S(T)}(t) = e^{-r\tau} E\left[s e^{(r-\frac{\sigma^2}{2})\tau + \sigma\sqrt{\tau}G} \right]$$

$$= s e^{-\frac{\sigma^2}{2}\tau} E\left[e^{\sigma\sqrt{\tau}G} \right] = s e^{-\frac{\sigma^2}{2}\tau} e^{\frac{\sigma^2}{2}\tau} = s.$$

Moreover,

$$\Pi_{\frac{1}{S(T)}}(t) = e^{-r\tau} E\left[\frac{1}{s e^{(r-\frac{\sigma^2}{2})\tau + \sigma\sqrt{\tau}G}} \right]$$

$$= e^{-r\tau} \frac{e^{-(r-\frac{\sigma^2}{2})\tau}}{s} E\left[e^{-\sigma\sqrt{\tau}G} \right]$$

$$= \frac{e^{-(2r-\frac{\sigma^2}{2})\tau}}{s} e^{\frac{1}{2}\sigma^2\tau} = \frac{1}{s} e^{(\sigma^2 - 2r)\tau}$$

and it follows that

$$\Pi_Y(t) = S(t) + \frac{1}{S(t)} e^{(\sigma^2 - 2r)\tau}.$$

2. Suppose $T > 0$, $N \in \mathbf{N}_+$, $h = \frac{T}{N}$, and $t_n = nh$, $n = 0, ..., N$, and consider a European-style derivative paying the amount $Y = \sum_{n=0}^{N-1} (\ln \frac{S(t_{n+1})}{S(t_n)})^2$ at time of maturity T. Find $\Pi_Y(0)$.

Solution. First consider a derivative paying the amount $Y_n = (\ln \frac{S(t_{n+1})}{S(t_n)})^2$ at time T. Since Y_n is known at time t_{n+1}, $\Pi_{Y_n}(t_{n+1}) = Y_n e^{-r(T-t_{n+1})}$. Note that

$$S(t_{n+1}) = S(t_n) e^{(\mu - \frac{\sigma^2}{2})h + \sigma(W(t_{n+1}) - W(t_n))}$$

where $W(t_{n+1}) - W(t_n) \in N(0, h)$. Thus, if $G \in N(0, 1)$,

$$\Pi_{Y_n}(t_n) = e^{-rh} E\left[e^{-r(T-t_{n+1})} \left\{ (r - \frac{\sigma^2}{2})h + \sigma\sqrt{h}G \right\}^2 \right]$$

$$= e^{-r(T-t_n)} \left\{ (r - \frac{\sigma^2}{2})^2 h^2 + \sigma^2 h \right\}$$

and since the expression for $\Pi_{Y_n}(t_n)$ is known at time 0,

$$\Pi_{Y_n}(0) = e^{-t_n r} e^{-r(T-t_n)} \left\{ (r - \frac{\sigma^2}{2})^2 h^2 + \sigma^2 h \right\}$$

$$= e^{-rT} \left\{ (r - \frac{\sigma^2}{2})^2 h^2 + \sigma^2 h \right\}.$$

Now it follows that

$$\Pi_Y(0) = \sum_{n=0}^{N-1} \Pi_{Y_n}(0) = N e^{-rT} \left\{ (r - \frac{\sigma^2}{2})^2 h^2 + \sigma^2 h \right\}$$

$$= T e^{-r\tau} \left\{ \sigma^2 + h(r - \frac{\sigma^2}{2})^2 \right\}.$$

3. A European-style derivative pays the amount

$$Y = 1 + S(T) \ln S(T)$$

at time of maturity T. (a) Find $\Pi_Y(t)$. (b) Find a hedging portfolio of the derivative at time t.

Solution. (a) If $s = S(t)$, $\tau = T - t$, and $G \in N(0,1)$, then

$$\Pi_Y(t) = e^{-r\tau} E\left[1 + se^{(r-\frac{\sigma^2}{2})\tau+\sigma\sqrt{\tau}G}\left\{\ln s + (r - \frac{\sigma^2}{2})\tau + \sigma\sqrt{\tau}G\right\}\right]$$

$$= e^{-r\tau} + s\left\{\ln s + (r - \frac{\sigma^2}{2})\tau\right\} e^{-\frac{\sigma^2}{2}\tau} E\left[e^{\sigma\sqrt{\tau}G}\right] + s\sigma\sqrt{\tau}E\left[Ge^{-\frac{\sigma^2}{2}\tau+\sigma\sqrt{\tau}G}\right]$$

$$= e^{-r\tau} + s\left\{\ln s + (r - \frac{\sigma^2}{2})\tau\right\} + s\sigma\sqrt{\tau}\int_{-\infty}^{\infty} xe^{-\frac{\sigma^2}{2}\tau+\sigma\sqrt{\tau}x-\frac{x^2}{2}}\frac{dx}{\sqrt{2\pi}}$$

$$= e^{-r\tau} + s\left\{\ln s + (r - \frac{\sigma^2}{2})\tau\right\} + s\sigma\sqrt{\tau}\int_{-\infty}^{\infty} xe^{-\frac{(x-\sigma\sqrt{\tau})^2}{2}}\frac{dx}{\sqrt{2\pi}} =$$

$$= e^{-r\tau} + s\left\{\ln s + (r - \frac{\sigma^2}{2})\tau\right\} + s\sigma\sqrt{\tau}\int_{-\infty}^{\infty} (y + \sigma\sqrt{\tau})e^{-\frac{y^2}{2}}\frac{dx}{\sqrt{2\pi}}$$

$$= e^{-r\tau} + s\left\{\ln s + (r - \frac{\sigma^2}{2})\tau\right\} + s\sigma^2\tau$$

$$= e^{-r\tau} + s\ln s + s(r + \frac{\sigma^2}{2})\tau$$

$$= e^{-r\tau} + S(t)\ln S(t) + S(t)(r + \frac{\sigma^2}{2})\tau$$

(b) A portfolio with

$$h_S(t) = \left(\frac{\partial}{\partial s}\left\{e^{-r\tau} + s\ln s + s(r + \frac{\sigma^2}{2})\tau\right\}\right)_{|s=S(t)}$$

$$= 1 + (r + \frac{\sigma^2}{2})\tau + \ln S(t)$$

units of the stock and

$$h_B(t)$$

$$= (e^{-r\tau} + S(t)\ln S(t) + S(t)(r + \frac{\sigma^2}{2})\tau - S(t)(1 + (r + \frac{\sigma^2}{2})\tau + \ln S(t)))/B(t)$$

$$= (e^{-r\tau} - S(t))/B(t)$$

units of the bond is a hedging portfolio at time t.

4. Suppose $0 < t_0 < T$ and $K > 0$. A financial European-style derivative pays the amount $Y = (\frac{S(T)}{S(t_0)} - K)^+$ at time of maturity T. Find the delta of the option at time t if (a) $0 < t < t_0$ (b) $t_0 < t < T$.

Solution. We first solve Part (b). Note that

$$Y = \frac{1}{S(t_0)}(S(T) - KS(t_0))^+$$

and, accordingly from this, if $t_0 \le t < T$,

$$\Pi_Y(t) = \frac{1}{S(t_0)}c(t, S(t), KS(t_0), T)$$

$$= \frac{1}{S(t_0)}\left\{S(t)\Phi(d_1(t)) - KS(t_0)e^{-r(T-t)}\Phi(d_2(t))\right\}$$

where

$$d_1(t) = \frac{\ln\frac{S(t)}{KS(t_0)} + (r + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}$$

and

$$d_2(t) = \frac{\ln\frac{S(t)}{KS(t_0)} + (r - \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}.$$

In particular,

$$\Pi_Y(t_0)$$

$$= \Phi(\frac{-\ln K + (r + \frac{\sigma^2}{2})(T-t_0)}{\sigma\sqrt{T-t_0}}) - Ke^{-r(T-t_0)}\Phi(\frac{-\ln K + (r - \frac{\sigma^2}{2})(T-t_0)}{\sigma\sqrt{T-t_0}})$$

and, moreover, from the known delta of a European call we get

$$\Delta(t) = \frac{1}{S(t_0)}\Phi(\frac{\ln\frac{S(t)}{KS(t_0)} + (r + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}), \text{ if } t_0 < t < T.$$

We next treat Part (a). If $s = S(t)$ and $0 < t < t_0$,

$$\Pi_Y(t) = e^{-r(t_0-t)}\Pi_Y(t_0)$$

since $\Pi_Y(t_0)$ is known at time t. Moreover, $\Pi_Y(t)$ is independent of s and we have

$$\Delta(t) = 0, \text{ if } 0 < t < t_0.$$

5. Suppose $0 < a < K$. A financial European-style derivative has the payoff $Y = g(S(T))$ at time of maturity T, where $g(x) =\mid x - K \mid$ if $x \notin \,]K-a, K+a[$ and $g(x) = 0$ if $x \in \,]K-a, K+a[$. Find a hedging portfolio for the derivative.

Solution. If $A \subseteq \,]0, \infty[$,

$$1_A(x) = \begin{cases} 1 \text{ if } x \in A \\ 0 \text{ if } x \notin A. \end{cases}$$

Now

$$g(x) = (K-a-x)^+ + a1_{]0,K-a]}(x) + (x-K-a)^+ + a1_{[K+a,\infty[}(x)$$

$$= K - a - x + (x-K+a)^+ + a - a1_{]K-a,\infty[}(x)$$

$$+(x-K-a)^+ + a1_{[K+a,\infty[}(x).$$

$$= K - x + (x-K+a)^+ - a1_{]K-a,\infty[}(x)$$

$$+(x-K-a)^+ + a1_{[K+a,\infty[}(x).$$

Hence, if $\tau = T - t > 0$, $s = S(t)$, and $G \in N(0,1), v(t,s) =_{def} \Pi_Y(t) = Ke^{-r\tau} - s + c(t,s,K-a,T)$

$$-ae^{-r\tau}E\left[1_{]K-a,\infty[}(se^{(r-\frac{\sigma^2}{2})\tau-\sigma\sqrt{\tau}G})\right] + c(t,s,K+a,T)$$

$$+ae^{-r\tau}E\left[1_{[K+a,\infty[}(se^{(r-\frac{\sigma^2}{2})\tau-\sigma\sqrt{\tau}G})\right]$$

$$= Ke^{-r\tau} - s + c(t,s,K-a,T) + c(t,s,K+a,T)$$

$$-ae^{-r\tau}\int_{x<\frac{\ln\frac{s}{K-a}+(r-\frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}}}\varphi(x)dx+ae^{-r\tau}\int_{x\leq\frac{\ln\frac{s}{K+a}+(r-\frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}}}\varphi(x)dx$$

$$=Ke^{-r\tau}-s+c(t,s,K-a,T)+c(t,s,K+a,T)-ae^{-r\tau}\Phi(\frac{\ln\frac{s}{K-a}+(r-\frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}})$$

$$+ae^{-r\tau}\Phi(\frac{\ln\frac{s}{K+a}+(r-\frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}}).$$

Recall that

$$c(t,s,K,T)=s\Phi(\frac{\ln\frac{s}{K}+(r+\frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}})-Ke^{-r\tau}\Phi(\frac{\ln\frac{s}{K}+(r-\frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}})$$

and

$$\frac{\partial c}{\partial s}=\Phi(\frac{\ln\frac{s}{K}+(r+\frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}}).$$

Now

$$h_S(t)=\frac{\partial v}{\partial s}=-1+\Phi(\frac{\ln\frac{s}{K-a}+(r+\frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}})+\Phi(\frac{\ln\frac{s}{K+a}+(r+\frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}})$$

$$-\frac{ae^{-r\tau}}{s\sigma\sqrt{\tau}}\varphi(\frac{\ln\frac{s}{K-a}+(r-\frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}})+\frac{ae^{-r\tau}}{s\sigma\sqrt{\tau}}\varphi(\frac{\ln\frac{s}{K+a}+(r-\frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}})$$

where, as said above, $s=S(t)$. Moreover,

$$h_B(t)=\frac{v(t,s)-h_S(t)S(t)}{B(t)}.$$

6. Suppose $0<a<b$ and $0\leq t<T$. A financial European-style derivative pays the amount Y at time of maturity T, where

$$Y=\begin{cases}1 \text{ if } S(T)\in\,]a,b[\,,\\ 0 \text{ if } S(T)\notin\,]a,b[\,.\end{cases}$$

(a) Find $\Pi_Y(t)$. (b) For which value on $S(t)$ is $\Pi_Y(t)$ maximal.

Solution. (a) Let $H(x) =$

$$H_0(x) = \begin{cases} 1 \text{ if } x > 0, \\ 0 \text{ if } x \le 0 \end{cases} \quad \text{and } H_1(x) = \begin{cases} 1 \text{ if } x \ge 0, \\ 0 \text{ if } x < 0. \end{cases}$$

Then

$$Y = H_0(S(T) - a) - H_1(S(T) - b).$$

Moreover, if $s = S(0)$ and $\tau = T - t$,

$$\Pi_{H_0(S(T)-a)}(t) = e^{-r\tau} \int_{-\infty}^{\infty} H_0(se^{(r-\frac{\sigma^2}{2})\tau - \sigma\sqrt{\tau}x} - a)\varphi(x)dx =$$

$$e^{-r\tau} \int_{-\infty}^{\frac{1}{\sigma\sqrt{\tau}}(\ln\frac{s}{a} + (r - \frac{\sigma^2}{2})\tau)} \varphi(x)dx = e^{-r\tau}\Phi(\frac{1}{\sigma\sqrt{\tau}}(\ln\frac{s}{a} + (r - \frac{\sigma^2}{2})\tau))$$

and, in a similar way,

$$\Pi_{H_1(S(T)-b)}(t) = e^{-r\tau}\Phi(\frac{1}{\sigma\sqrt{\tau}}(\ln\frac{s}{b} + (r - \frac{\sigma^2}{2})\tau)).$$

Thus

$$\Pi_Y(t) =$$

$$e^{-r\tau}\left(\Phi(\frac{1}{\sigma\sqrt{\tau}}(\ln\frac{s}{a} + (r - \frac{\sigma^2}{2})\tau)) - \Phi(\frac{1}{\sigma\sqrt{\tau}}(\ln\frac{s}{b} + (r - \frac{\sigma^2}{2})\tau))\right).$$

(b) Set $\Pi_Y(t) = v(s)$. Since $\frac{s}{a} > \frac{s}{b}$ and Φ is strictly increasing, it is obvious that v is a positive function. Moreover, v is continuous and

$$\lim_{s\to\infty} v(s) = \lim_{s\to 0+} v(s) = 0.$$

From this we conclude thar v attains a maximum and the derivative of $v(s)$ vanishes at this point.

We have

$$v'(s) =$$

$$\frac{e^{-r\tau}}{s\sigma\sqrt{\tau}}\left(\varphi(\frac{1}{\sigma\sqrt{\tau}}(\ln\frac{s}{a} + (r - \frac{\sigma^2}{2})\tau)) - \varphi(\frac{1}{\sigma\sqrt{\tau}}(\ln\frac{s}{b} + (r - \frac{\sigma^2}{2})\tau))\right)$$

and, hence $v'(s) = 0$ if and only if

$$(\ln\frac{s}{a} + (r - \frac{\sigma^2}{2})\tau)^2 = (\ln\frac{s}{b} + (r - \frac{\sigma^2}{2})\tau))^2.$$

Thus

$$\ln\frac{s}{a}+(r-\frac{\sigma^2}{2})\tau=\pm(\ln\frac{s}{b}+(r-\frac{\sigma^2}{2})\tau).$$

Here the plus sign leads to $a=b$, which is a contradiction, and we must have

$$2\ln s=\ln ab-2(r-\frac{\sigma^2}{2})\tau$$

or

$$s=\sqrt{ab}e^{-(r-\frac{\sigma^2}{2})\tau}.$$

7. Suppose $K,T>0$ are constants. A financial European-style derivative has the payoff

$$Y=\begin{cases}1 \text{ if } S(T)>K,\\ -1 \text{ if } S(T)\le K,\end{cases}$$

at time of maturity T. Determine K such that $\Pi_Y(0)=0$.

Solution. Put

$$g(x)=\begin{cases}1 \text{ if } x>K,\\ -1 \text{ if } x\le K,\end{cases}$$

and note that $Y=g(S(T))$. Now

$$\Pi_Y(0)=e^{-rT}E\left[g(se^{(r-\frac{\sigma^2}{2})T+\sigma\sqrt{T}G})\right]$$

where $s=S(0)$ and $G\in N(0,1)$ and, hence,

$$\Pi_Y(0)=e^{-rT}\left(P\left[se^{(r-\frac{\sigma^2}{2})T+\sigma\sqrt{T}G}>K\right]-P\left[se^{(r-\frac{\sigma^2}{2})T+\sigma\sqrt{T}G}\le K\right]\right)$$

$$=e^{-rT}\left(2P\left[se^{(r-\frac{\sigma^2}{2})T+\sigma\sqrt{T}G}>K\right]-1\right)$$

$$=e^{-rT}\left(2P\left[se^{(r-\frac{\sigma^2}{2})T-\sigma\sqrt{T}G}>K\right]-1\right)$$

$$=e^{-rT}\left(2P\left[G\le\frac{1}{\sigma\sqrt{T}}(\ln\frac{s}{K}+(r-\frac{\sigma^2}{2})T)\right]-1\right)$$

$$=e^{-rT}\left(2\Phi(\frac{1}{\sigma\sqrt{T}}(\ln\frac{s}{K}+(r-\frac{\sigma^2}{2})T))-1\right).$$

Accordingly from this, $\Pi_Y(0) = 0$ if and only if

$$\ln \frac{s}{K} + (r - \frac{\sigma^2}{2})T = 0$$

that is,

$$K = S(0)e^{(r-\frac{\sigma^2}{2})T}.$$

8. Suppose $T > 0$. A financial European-style derivative pays the amount

$$Y = \max(\frac{S(\frac{T}{2})}{S(0)}, \frac{S(T)}{S(\frac{T}{2})})$$

at time of maturity T. Find $\Pi_Y(0)$.

Proof. Put $S(0) = s$ and $a = \frac{T}{2}$. We have

$$\Pi_Y(0) = e^{-rT} E\left[\max(\frac{se^{ra+\sigma W(a)}}{s}, \frac{se^{rT+\sigma W(T)}}{se^{ra+\sigma W(a)}})\right]$$

$$= e^{-rT} E\left[\max(e^{ra+\sigma W(a)}, e^{ra+\sigma(W(T)-W(a))})\right]$$

$$= e^{-ra} E\left[\max(e^{\sigma\sqrt{a}G}, e^{\sigma\sqrt{a}H})\right]$$

$$= e^{-ra} E\left[e^{\sigma\sqrt{a}\max(G,H)}\right],$$

where $G, H \in N(0,1)$ are independent. Moreover,

$$P\left[\max(G,H) \leq x\right] = P\left[G \leq x,\ H \leq x\right]$$

$$= P\left[G \leq x\right] P\left[\ H \leq x\right] = \Phi^2(x).$$

Hence

$$E\left[e^{\sigma\sqrt{a}\max(G,H)}\right] = \int_{-\infty}^{\infty} e^{\sigma\sqrt{a}x} \frac{d}{dx}\Phi^2(x)dx$$

$$= 2\int_{-\infty}^{\infty} e^{\sigma\sqrt{a}x}\Phi(x)\varphi(x)dx.$$

Now introduce $b = \sigma\sqrt{a}$ and note that

$$\int_{-\infty}^{\infty} e^{bx}\Phi(x)\varphi(x)dx = e^{\frac{b^2}{2}} \int_{-\infty}^{\infty} \Phi(x)\varphi(x-b)dx$$

$$= e^{\frac{b^2}{2}} \int_{-\infty}^{\infty} \Phi(b-x)\varphi(x)dx.$$

But

$$\int_{-\infty}^{\infty} \varphi(y-x)\varphi(x)dx = \frac{1}{\sqrt{2}}\varphi(\frac{y}{\sqrt{2}})$$

since $G + H \in N(0,2)$ and by integration from $y = -\infty$ to $y = b$ we get

$$\int_{-\infty}^{\infty} \Phi(b-x)\varphi(x)dx = \int_{-\infty}^{b} \frac{1}{\sqrt{2}}\varphi(\frac{y}{\sqrt{2}})dy = \Phi(\frac{b}{\sqrt{2}}).$$

Hence

$$\int_{-\infty}^{\infty} e^{bx}\Phi(x)\varphi(x)dx = e^{\frac{b^2}{2}}\Phi(\frac{b}{\sqrt{2}})$$

and

$$\Pi_Y(0) = 2e^{-ra}e^{\frac{\sigma^2 a}{2}}\Phi(\frac{\sigma\sqrt{a}}{\sqrt{2}})$$

$$= 2e^{-\frac{rT}{2}}e^{\frac{\sigma^2 T}{4}}\Phi(\frac{\sigma\sqrt{T}}{2}).$$

CHAPTER 6

Several sources of randomness

Introduction

To begin with in this chapter we are going to price the option on the maximum of two stock prices and the option to exchange one stock for another. Furthermore, we will price a call on a foreign equity, struck in domestic currency. In both cases we will proceed in Black-Scholes like models.

Besides this chapter will illustrate how several sources of randomness and high-dimensional Brownian motion enter in various multi-asset models of standard type.

6.1 Bivariate Geometric Brownian Motion

We start with two important definitions. Two random vectors $X = (X_1, ..., X_m) = (X_k)_{k=1}^m$ and $Y = (Y_1, ..., Y_n) = (Y_k)_{k=1}^n$,in $\mathbf{R}^m$ and $\mathbf{R}^n$, respectively, are said to be independent if

$$E[f(X)g(Y)] = E[f(X)]\,E[g(Y)]$$

for all continuous functions $f : \mathbf{R}^m \to \mathbf{R}$ and $g : \mathbf{R}^n \to \mathbf{R}$ such that $E[\mid f(X) \mid] < \infty$ and $E[\mid g(Y) \mid] < \infty$ (for $n = 1$ it can be proved that this definition is equivalent to the definition given in Chapter 3). Moreover, two stochastic processes $X = (X(t))_{t \in T}$ and $Y = (Y(u))_{u \in U}$ are said to be independent if $(X(t))_{t \in T_0}$ and $(Y(u))_{u \in U_0}$ are independent for every finite subset T_0 of T and every finite subset U_0 of U.

Throughout this section $Z_1 = (Z_1(t))_{t \geq 0}$ and $Z_2 = (Z_2(t))_{t \geq 0}$ denote two independent standard Brownian motions with continuous sample paths and $Z(t) = (Z_1(t), Z_2(t))$, $t \geq 0$. The process $Z = (Z(t))_{t \geq 0}$ is called a standard Brownian motion in the plane.

If $a = (a_1, a_2)$ and $b = (b_1, b_2)$ are vectors in the plane, let $a \cdot b = a_1 b_1 + a_2 b_2$ be the dot product of a and b and $\mid a \mid = \sqrt{a \cdot a}$ the length of a. A vector in $\mathbf{R}^2$ is often identified with a column matrix.

Theorem 6.1.1. *Suppose the vectors* $a_i = (a_{i1}, a_{i2}) \in \mathbf{R}^2$, $i = 1, 2,$ *and let* $A = (a_{ik})_{1 \le i,k \le 2}$. *Moreover, set* $X = AZ$, *that is,* $X(t) = (X_1(t), X_2(t))_{t \ge 0}$, *where*

$$\begin{aligned} X_1(t) &= a_{11} Z_1(t) + a_{12} Z_2(t) \\ X_2(t) &= a_{21} Z_1(t) + a_{22} Z_2(t). \end{aligned}$$

Then

$$E\left[X_i(t)\right] = 0,\ \ i = 1, 2$$

$$\mathrm{Var}(X_i(t)) = \mid a_i \mid^2 t,\ \ i = 1, 2$$

and

$$\mathrm{Cov}(X_1(s), X_2(t)) = (a_1 \cdot a_2) \min(s, t).$$

.

PROOF Let $i = 1$ or 2. Then $E\left[X_i(t)\right] = a_{i1} E\left[Z_1(t)\right] + a_{i2} E\left[Z_2(t)\right] = 0$ and

$$\mathrm{Cov}(X_1(s), X_2(t)) = E\left[X_1(s) X_2(t)\right]$$

$$= a_{11} a_{21} E\left[Z_1(s) Z_1(t)\right] + a_{11} a_{22} E\left[Z_1(s) Z_2(t)\right]$$

$$+ a_{12} a_{21} E\left[Z_2(s) Z_1(t)\right] + a_{12} a_{22} E\left[Z_2(s) Z_2(t)\right]$$

$$= a_{11} a_{21} \min(s, t) + a_{12} a_{22} \min(s, t) = (a_1 \cdot a_2) \min(s, t)$$

since $E\left[Z_1(s) Z_2(t)\right] = E\left[Z_1(s)\right] E\left[Z_2(t)\right] = 0$. In particular, if $i = 1$ or 2,

$$\mathrm{Var}(X_i(t)) = E\left[X_i^2(t)\right] = \mid a_i \mid^2 t.$$

This proves Theorem 6.1.1.

Corollary 6.1. *If a is a unit vector in the plane, $a \cdot Z$ is a real-valued standard Brownian motion.*

Suppose two stock price processes $S_1 = (S_1(t))_{t\geq 0}$ and $S_2 = (S_2(t))_{t\geq 0}$ are governed by geometric Brownian motions. The stock with the price process $(S_i(t))_{t\geq 0}$ will be called the $i : th$ stock for $i = 1, 2$. Thus, by assumption, there are standard Brownian motions $(W_1(t))_{t\geq 0}$ and $(W_2(t))_{t\geq 0}$ such that

$$S_1(t) = S_1(0)e^{\alpha_1 t + \sigma_1 W_1(t)}$$

and

$$S_2(t) = S_2(0)e^{\alpha_2 t + \sigma_2 W_2(t)}$$

where $\alpha_1, \alpha_2 \in \mathbf{R}$ and $\sigma_1, \sigma_2 > 0$. In particular,

$$E\left[W_1(t)\right] = E\left[W_2(t)\right] = 0$$

and

$$E\left[W_1(s)W_1(t)\right] = E\left[W_2(s)W_2(t)\right] = \min(s,t).$$

To define a correlation between the stock price process, we assume there is a standard Brownian motion $Z = (Z_1, Z_2) = (Z_1(t), Z_2(t))_{t\geq 0}$ in the plane and a real number $\rho \in \,]-1, 1[$ such that

$$\begin{cases} W_1(t) = Z_1(t) \\ W_2(t) = \rho Z_1(t) + \sqrt{1-\rho^2}\, Z_2(t). \end{cases}$$

Under all these assumptions the joint stock price process $S = (S_1(t), S_2(t))_{t\geq 0}$ is called a bivariate geometric Brownian motion with volatility (σ_1, σ_2) and correlation ρ. If so, it is readily seen that the process $(S_2(t), S_1(t))_{t\geq 0}$ is a bivariate geometric Brownian motion with volatility (σ_2, σ_1) and correlation ρ.

In general, we do not have the mathematical machinery required to price options on (S_1, S_2) but there are some important special cases, which may be reduced to options on one stock by assuming appropriate conditions on the payoff.

Suppose $g \in \mathcal{P}_2$ is positively homogenous of degree one, that is

$$g(\lambda x_1, \lambda x_2) = \lambda g(x_1, x_2), \;\; \lambda > 0$$

and consider an European option on (S_1, S_2) with the payoff

$$g(S_1(T), S_2(T))$$

at the termination date T. To find a natural price of this derivative we take stock 2 as a numéraire and denominate the first stock in terms of the chosen numéraire. In terms of the new numéraire stock 1 and 2 get the price processes

$$S(t) = \frac{S_1(t)}{S_2(t)},\ t \geq 0$$

and

$$B(t) = 1,\ t \geq 0.$$

respectively, and, in addition, the option pays $f(S(T))$ units of stock 2 at the termination date T, where

$$f(x) = g(x, 1)$$

as

$$g(x_1, x_2) = x_2 g(\frac{x_1}{x_2}, 1).$$

Moreover, setting

$$\sigma_- = \sqrt{\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2}$$

we have

$$S(t) = S(0)e^{(\alpha_1 - \alpha_2)t + \sigma_- W_-(t)}.$$

Here, in view of Theorem 6.1.1, the process

$$W_-(t) = \frac{1}{\sigma_-}\left\{(\sigma_1 - \rho\sigma_2)Z_1(t) - \sigma_2\sqrt{1-\rho^2}Z_2(t)\right\},\ t \geq 0$$

is a standard Brownian motion in **R**. Thus we are in a Black-Schole like model with interest rate zero and a natural price of the option at time t equals $v(t, S(t))$ where

$$v(t, s) = E\left[f(se^{-\frac{\sigma_-^2}{2}\tau + \sigma_-\sqrt{\tau}G})\right]$$

and $\tau = T - t$ is the residual time. Thus in the original price unit the option price at time t equals $u(t, S_1(t), S_2(t))$ with

$$u(t, s_1, s_2) = s_2 v(t, \frac{s_1}{s_2}).$$

In particular, no bond is needed for the pricing of the option or, stated otherwise, the option price is (explicitely) independent of the interest rate.

If $t < T$ iterated differentiation yields the following derivatives

$$\frac{\partial u}{\partial s_1} = \frac{\partial v}{\partial s}(t, \frac{s_1}{s_2})$$

$$\frac{\partial^2 u}{\partial s_1^2} = \frac{1}{s_2}\frac{\partial^2 v}{\partial s^2}(t, \frac{s_1}{s_2})$$

$$\frac{\partial^2 u}{\partial s_1 \partial s_2} = -\frac{s_1}{s_2^2}\frac{\partial^2 v}{\partial s^2}(t, \frac{s_1}{s_2})$$

$$\frac{\partial u}{\partial s_2} = v(t, \frac{s_1}{s_2}) - \frac{s_1}{s_2}\frac{\partial v}{\partial s}(t, \frac{s_1}{s_2})$$

$$\frac{\partial^2 u}{\partial s_2^2} = \frac{s_1^2}{s_2^3}\frac{\partial^2 v}{\partial s^2}(t, \frac{s_1}{s_2})$$

and we conclude that

$$\sigma_1^2 s_1^2 \frac{\partial^2 u}{\partial s_1^2} + 2\rho\sigma_1\sigma_2 s_1 s_2 \frac{\partial^2 u}{\partial s_1 \partial s_2} + \sigma_2^2 s_2^2 \frac{\partial^2 u}{\partial s_2^2}$$

$$= s_2 \sigma^2 (\frac{s_1}{s_2})^2 \frac{\partial^2 v}{\partial s^2}(t, \frac{s_1}{s_2}).$$

In the limit case $r = 0$ the Black-Scholes differential equation reduces to

$$\frac{\partial v}{\partial t} + \frac{\sigma^2}{2} s^2 \frac{\partial^2 v}{\partial s^2} = 0$$

and it follows that

$$\frac{\partial u}{\partial t} + \frac{1}{2}\left\{\sigma_1^2 s_1^2 \frac{\partial^2 u}{\partial s_1^2} + 2\rho\sigma_1\sigma_2 s_1 s_2 \frac{\partial^2 u}{\partial s_1 \partial s_2} + \sigma_2^2 s_2^2 \frac{\partial^2 u}{\partial s_2^2}\right\} = 0.$$

In addition

$$u_{|t=T} = g.$$

The following example is due to Margrabe $[MAR]$.

Example 6.1.1. Consider the option to change stock 2 for stock 1 at time T. Now

$$g(x_1, x_2) = \max(0, x_1 - x_2)$$

and

$$f(x) = \max(0, x - 1).$$

Thus, $t < T$,

$$v(t, s) = s\Phi(d_1) - \Phi(d_2)$$

where

$$d_1(s) = \frac{\ln s + \frac{\sigma_-^2}{2}\tau}{\sigma_-\sqrt{\tau}}$$

and

$$d_2(s) = \frac{\ln s - \frac{\sigma_-^2}{2}\tau}{\sigma_-\sqrt{\tau}}.$$

At time t the option price equals $u(t, S_1(t), S_2(t))$, where

$$\begin{aligned} u(t, s_1, s_2) &= s_2 v(t, \frac{s_1}{s_2}) \\ &= s_1\Phi(d_1(\frac{s_1}{s_2})) - s_2\Phi(d_2(\frac{s_1}{s_2})). \end{aligned}$$

Example 6.1.2. Consider the European option on the maximum of the prices of stock 1 and stock 2 at time T. Now

$$g(x_1, x_2) = \max(x_1, x_2)$$

and, since

$$g(x_1, x_2) = x_2 + \max(0, x_1 - x_2)$$

we get

$$u(t, s_1, s_2) = s_1\Phi(d_1(\frac{s_1}{s_2})) + s_2(1 - \Phi(d_2(\frac{s_1}{s_2})))$$

which reduces to

$$u(t, s_1, s_2) = s_1\Phi(d_1(\frac{s_1}{s_2})) + s_2\Phi(d_1(\frac{s_2}{s_1})).$$

Example 6.1.3. Consider a European call on the US IBM stock with the price process $(U(t))_{t\geq 0}$ in US dollars with the termination date T and the strike price K Swedish crowns. We assume the Swedish market offers a bond with the price process

$$B = (B(0)e^{rt})_{t \geq 0}$$

where r is the positive constant interest rate. If $\xi = (\xi(t))_{t \geq 0}$ is the exchange rate Swedish crowns per US dollars, then at maturity the payoff is

$$(U(T)\xi(T) - K)^+ \text{ Swedish crowns.}$$

To price this contingent claim on the Swedish option market, we assume (U, ξ) is a bivariate geometric Brownian motion with volatility (σ_U, σ_ξ) and correlation ρ. Thus, let

$$U(t) = U(0)e^{\alpha_U t + \sigma_U W_U(t)}$$

and

$$\xi(t) = \xi(0)e^{\alpha_\xi t + \sigma_\xi W_\xi(t)}$$

where

$$\begin{cases} W_U(t) = Z_1(t) \\ W_\xi(t) = \rho Z_1(t) + \sqrt{1-\rho^2} Z_2(t) \end{cases}$$

and $\alpha_U, \alpha_\xi \in \mathbf{R}$, $\sigma_U, \sigma_\xi > 0$ $\rho \in \left]-1, 1\right[$. Here as above $Z = (Z_1, Z_2)$ is a standard Brownian motion in the plane.

The process

$$S(t) = U(t)\xi(t), \ t \geq 0$$

can be viewed as the price process of a traded Swedish security and

$$S(t) = S(0)e^{(\alpha_U + \alpha_\xi)t + \sigma_+ W_+(t)}$$

where

$$W_+(t) = \frac{1}{\sigma} \left\{ (\sigma_U + \rho\sigma_\xi) Z_1(t) + \sqrt{1-\rho^2} \sigma_\xi Z_2(t) \right\}$$

and

$$\sigma_+ = \sqrt{\sigma_U{}^2 + 2\rho\sigma_U\sigma_\xi + \sigma_\xi^2}.$$

Now since $(W_+(t))_{t \geq 0}$ is a standard Brownian motion, the Black Scholes call price applies and, if the residual time $\tau = T - t > 0$, we get the following option price in Swedish crowns, namely

$$U(t)\xi(t)\Phi(d_1) - Ke^{-r\tau}\Phi(d_2)$$

where

$$d_1 = \frac{\ln \frac{U(t)\xi(t)}{K} + (r + \frac{\sigma_+^2}{2})\tau}{\sigma_+\sqrt{\tau}}$$

and

$$d_2 = \frac{\ln \frac{U(t)\xi(t)}{K} + (r - \frac{\sigma_+^2}{2})\tau}{\sigma_+\sqrt{\tau}}.$$

To check the price let us consider a European exchange option on the US option market paying the amount

$$(U(T) - \frac{K}{B(T)}B(T)/\xi(T)) \text{ US dollars}$$

at the termination date T. Here the process

$$(\frac{K}{B(T)}B(t)/\xi(t))_{t\geq 0}$$

can be viewed as the price process of a traded US security. Still assuming that (U, ξ) is a bivariate geometric Brownian motion with volatility $\sigma = (\sigma_S, \sigma_\xi)$ and correlation ρ, $(U, 1/\xi)$ is a bivariate geometric Brownian motion with volatility (σ_S, σ_ξ) and correlation $-\rho$. Thus at time $t < T$, Example 6.1.2 says that the price in US dollars of the contract equals

$$U(t)\Phi(\frac{1}{\sigma_+\sqrt{\tau}}(\ln \frac{U(t)}{\frac{K}{B(T)}B(t)/\xi(t)} + \frac{\sigma_+^2}{2}\tau))$$

$$-\frac{K}{B(T)}B(t)\xi^{-1}(t)\Phi(\frac{1}{\sigma_+\sqrt{\tau}}(\ln \frac{U(t)}{\frac{K}{B(T)}B(t)/\xi(t)} - \frac{\sigma_+^2}{2}\tau)).$$

Finally, multiplying this expression by $\xi(t)$ to obtain the option price in Swedish crowns at time t, after some simplifications, we get the same price as above.

If U is as above, a European-style contingent claim which pays the amount

$$Y = \max(0, U(T) - K)$$

Swedish crowns at the termination date T cannot be handled by the methods in these lecture notes.

Exercises

1. Suppose $G_1, G_2 \in N(0,1)$ are independent and set

$$\begin{pmatrix} X_1 & X_2 \end{pmatrix} = \begin{pmatrix} G_1 & G_2 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$$

and

$$\begin{pmatrix} Y_1 & Y_2 \end{pmatrix} = \begin{pmatrix} G_1 & G_2 \end{pmatrix} \begin{pmatrix} \sqrt{2} & 3/\sqrt{2} \\ 0 & 1/\sqrt{2} \end{pmatrix}.$$

Show that the processes $(X_k)_{k=1}^2$ and $(Y_k)_{k=1}^2$ are equivalent in distribution.

2. Suppose $g \in \mathcal{P}_2$ is positively homogeneous of degree one and consider a European-style derivative with the payoff

$$Y = g(S_1(T), S_2(T))$$

at the termination date T, where the joint stock price process (S_1, S_2) is a bivariate geometric Brownian motion. Show that

$$\Pi_Y(t) \geq g(S_1(t), S_2(t))$$

if g is convex.

3. Suppose $Z = (Z_1(t), Z_2(t))_{t \geq 0}$ is a standard Brownian motion in the plane. Find

$$E\left[\mid Z_1(t) - Z_2(t) \mid e^{(Z_1(t)+Z_2(t))^2}\right] \text{ if } 0 \leq t < \frac{1}{4}.$$

(Answer: $2\sqrt{\frac{t}{\pi(1-4t)}}$)

6.2 A single-period model with $n+1$ assets

Next we will give an introduction to the well known Markowitz mean-variance approach to portfolio selection, which appeared in 1952. The presentation is rather brief and mainly serve as an example of an application of covariance analysis and high-dimentional probability (for applications to real stock market data, see e.g. $[DGU]$).

Consider a so called single-period model with $n+1$ strictly positive asset price processes. The time set consists of the points $t=0$ and $t=T$ and the asset price processes are denoted by

$$S_i = (S_i(t))_{t=0,T}, \ \ i = 1, ..., n, n+1.$$

As usual the return of the i:th asset is defined by

$$R_i = \frac{S_i(T) - S_i(0)}{S_i(0)}, \ \ i = 1, ..., n, n+1.$$

Here the first n assets are risky and the $(n+1)$:th asset is a bond with the corresponding return R_{n+1} which is known at time 0. To emphasize that R_{n+1} is deterministic we sometimes write $R_{n+1} = r$.

Next we consider an investor with wealth $K > 0$ at time zero, who wants to distribute this wealth among the $n+1$ assets. To this end suppose the amount α_i is invested in the i:th asset at time zero so that

$$\sum_{i=1}^{n+1} \alpha_i = K.$$

The number of shares invested in the i:th asset is equal to

$$a_i = \frac{\alpha_i}{S_i(0)}, \ \ i = 1, ..., n, n+1$$

and the corresponding fractions of the capital invested in the assets are given by the quantities

$$\pi_i = \frac{\alpha_i}{K} = \frac{a_i S_i(0)}{K}, \ \ i = 1, ..., n, n+1.$$

Note that

$$\sum_{i=1}^{n+1} \pi_i = 1.$$

Now if $V(t)$ denotes the portfolio value at time t it follows that

$$V(0) = K = \sum_{i=1}^{n+1} a_i S_i(0)$$

and

$$V(T) = \sum_{i=1}^{n+1} a_i S_i(T).$$

As

$$S_i(T) = S_i(0)(1 + R_i), \; i = 1, ..., n, n+1$$

the return of the portfolio equals

$$R = \frac{V(T) - V(0)}{V(0)} = \sum_{i=1}^{n+1} \frac{a_i S_i(0)}{K} R_i$$

and

$$R = \sum_{i=1}^{n+1} \pi_i R_i.$$

Moreover,

$$E[R] = \sum_{i=1}^{n+1} \pi_i E[R_i]$$

and, since $R_{n+1} = r = E[R_{n+1}]$,

$$R - E[R] = \sum_{i=1}^{n+1} \pi_i (R_i - E[R_i]) = \sum_{i=1}^{n} \pi_i (R_i - E[R_i]).$$

Thus, with

$$\mu_i = E[R_i], \; i = 1, ..., n$$

we have

$$\mathrm{Var}(R) = \sum_{i,j=1}^{n} \pi_i \pi_j E\left[(R_i - \mu_i)(R_j - \mu_j)\right]$$

or

$$\mathrm{Var}(R) = \sum_{i,j=1}^{n} \pi_i \pi_j c_{ij}$$

where

$$C = (c_{ij})_{1 \le i,j \le n} = (\mathrm{Cov}(R_i, R_j))_{1 \le i,j \le n}.$$

Note that $C = C^\intercal$, that is, the matrix C is symmetric. Throughout this chapter it will be assumed that the covariance matrix C is invertible and we write

$$C^{-1} = (c_{ij}^{-1})_{1 \le i,j \le n}.$$

In the so called mean-variance approach to portfolio selections in the above single-period model it is optimal to maximize the function

$$f(\pi_1, ..., \pi_{n+1}) = E[R] - \theta \mathrm{Var}(R)$$

under the constraint

$$\sum_{i=1}^{n+1} \pi_i = 1$$

where θ is a strictly positive constant, which quantifies the risk aversion of the investor. Solving for π_{n+1} the investor will maximize the function

$$g(\pi_1, ..., \pi_n) =_{def} f(\pi_1, ..., \pi_n, 1 - \sum_{i=1}^{n} \pi_i)$$

$$= r + \sum_{i=1}^{n} \pi_i(\mu_i - r) - \theta \sum_{i,j=1}^{n} \pi_i \pi_j c_{ij}$$

over all reals $\pi_1, ..., \pi_n$. It is customary to introduce

$$\gamma = 1 - 2\theta$$

so that $\gamma < 1$ and, hence

$$g(\pi_1, ..., \pi_n)$$

$$= r + \sum_{i=1}^{n} \pi_i(\mu_i - r) - \frac{1-\gamma}{2} \sum_{i,j=1}^{n} \pi_i \pi_j c_{ij}.$$

As

$$\frac{\partial g}{\partial \pi_i} = \mu_i - r - (1-\gamma) \sum_{j=1}^{n} \pi_j c_{ij}, \ i = 1, ..., n$$

the equations

$$\frac{\partial g}{\partial \pi_i} = 0, \ i = 1, ..., n$$

yield

$$\begin{bmatrix} \pi_1 \\ \vdots \\ \pi_n \end{bmatrix} = \frac{C^{-1}}{1-\gamma} \begin{bmatrix} \mu_1 - r \\ \vdots \\ \mu_n - r \end{bmatrix}.$$

It turns out that this necessary condition for a maximum is also a sufficient condition. In fact, introduce

$$\begin{bmatrix} \hat{\pi}_1 \\ \vdots \\ \hat{\pi}_n \end{bmatrix} =_{def} \frac{C^{-1}}{1-\gamma} \begin{bmatrix} \mu_1 - r \\ \vdots \\ \mu_n - r \end{bmatrix}$$

and let

$$g(\pi_1, ..., \pi_n) = g(\hat{\pi}_1 + (\pi_1 - \hat{\pi}_1), ..., \hat{\pi}_n + (\pi_n - \hat{\pi}_n))$$

$$= a + \sum_1^n b_i(\pi_i - \hat{\pi}_i) - \frac{1-\gamma}{2} \sum_{i,j=1}^n (\pi_i - \hat{\pi}_i)(\pi_j - \hat{\pi}_j)c_{ij}$$

for appropriate real numbers $a, b_1, ..., b_n$. Here, clearly

$$a = g(\hat{\pi}_1, ..., \hat{\pi}_n)$$

and

$$b_i = \frac{\partial g}{\partial \pi_i}(\hat{\pi}_1, ..., \hat{\pi}_n), \ i = 1, .., n.$$

Thus $b_1 = ... = b_n = 0$ and we get

$$g(\pi_1, ..., \pi_n) \leq g(\hat{\pi}_1, ..., \hat{\pi}_n)$$

since

$$\sum_{i,j=1}^n (\pi_i - \hat{\pi}_i)(\pi_j - \hat{\pi}_j)c_{ij} = \mathrm{Var}(\sum_{i=1}^n (\pi_i - \hat{\pi}_i)R_i) \geq 0.$$

Next let

$$\hat{\pi}' = \begin{bmatrix} \hat{\pi}_1 \\ \vdots \\ \hat{\pi}_n \end{bmatrix},$$

$$\mu = \begin{bmatrix} \mu_1 \\ \vdots \\ \mu_n \end{bmatrix}$$

and introduce the following n by 1 matrix

$$1_n = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}.$$

From now on we assume

$$1_n^\intercal \hat{\pi}' \neq 0.$$

Stated otherwise, as

$$\hat{\pi}' = \frac{C^{-1}}{1-\gamma}(\mu - r1_n)$$

we assume

$$\delta =_{def} 1_n^\intercal C^{-1}(\mu - r1_n) \neq 0$$

or

$$r \neq \frac{1_n^\intercal C^{-1}\mu}{1_n^\intercal C^{-1}1_n}.$$

Definition 6.2.1. The Markowitz portfolio π_M is an $n+1$ by 1 matrix, where the last row is zero and the remaining first rows are given by the n by 1 matrix $\frac{1}{\delta}C^{-1}(\mu - r1_n)$. The latter matrix is denoted by π'_M. Thus

$$\pi'_M = \frac{1}{\delta}C^{-1}(\mu - r1_n).$$

Note that the vector π'_M satisfies the equation $(\pi'_M)^\intercal 1_n = 1$.

Theorem 6.2.1. (Tobin's Mutual Fund Theorem) *The optimal portfolio*

$$\hat{\pi} =_{def} \begin{bmatrix} \hat{\pi}_1 \\ \vdots \\ \hat{\pi}_n \\ \hat{\pi}_{n+1} \end{bmatrix}$$

is given by the equation

$$\hat{\pi} = \frac{1}{1-\gamma}\begin{bmatrix} \sum_{j=1}^n c_{1j}^{-1}(\mu_j - r) \\ \vdots \\ \sum_{j=1}^n c_{nj}^{-1}(\mu_j - r) \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 - \sum_{i=1}^n \hat{\pi}_i \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\delta}{1-\gamma}\pi'_M \\ 1 - \frac{1}{1-\gamma}1_n C^{-1}(\mu - r1_n) \end{bmatrix}.$$

The mutual fund theorem implies that regardless of the risk aversion parameter θ the investor will mix the Markowitz portfolio and the bond in appropriate proportions to get the optimal portfolio selection.

Example 6.2.1. (The Capital Asset Pricing Model (CAPM)) Let

$$R_M = (\pi_M)^\intercal \begin{bmatrix} R_1 \\ \vdots \\ R_{n+1} \end{bmatrix} = (\pi'_M)^\intercal \begin{bmatrix} R_1 \\ \vdots \\ R_n \end{bmatrix}$$

be the return of the Markowitz portfolio. Solving for μ in the definition of π'_M in Definition 6.2.1 yields

$$\mu = \delta C \pi'_M + r 1_n$$

and we have

$$\mu_M =_{def} E\,[R_M] = (\pi'_M)^\intercal \mu$$

$$= (\pi'_M)^\intercal (\delta C \pi'_M + r 1_n) = \delta \mathrm{Var}(R_M) + r.$$

Moreover, if $e_1, ..., e_n$ denotes the standard basis in $\mathbf{R}^n$ (= the space of all n by 1 matrices with real entries),

$$\mu_i = e_i^\intercal \mu = e_i^\intercal (\delta C \pi'_M + r 1_n)$$

$$= \delta e_i^\intercal C \pi'_M + r = \delta \mathrm{Cov}(R_i, R_M) + r, \; i = 1, ..., n.$$

Hence

$$\frac{\mathrm{Cov}(R_i, R_M)}{\mathrm{Var}(R_M)} = \frac{\mu_i - r}{\mu_M - r}, \; i = 1, ..., n$$

or

$$\mu_i - r = \frac{\mathrm{Cov}(R_i, R_M)}{\mathrm{Var}(R_M)} (\mu_M - r), \; i = 1, ..., n.$$

Now defining

$$\beta_i = \frac{\mathrm{Cov}(R_i, R_M)}{\mathrm{Var}(R_M)}, \; i = 1, ..., n$$

we have the remarkable relations

$$\mu_i - r = \beta_i(\mu_M - r),\ i = 1, ..., n$$

and, in particular,

$$\hat{\pi}' = \begin{bmatrix} \hat{\pi}_1 \\ \vdots \\ \hat{\pi}_n \end{bmatrix} = \frac{\mu_M - r}{1-\gamma} C^{-1} \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_n \end{bmatrix}.$$

Exercises

1. Prove that $\hat{\pi}' \to 0$ as $\gamma \downarrow -\infty$.

2. Let A be a symmetric invertible n by n matrix such that $x^\intercal A x \geq 0$ if x is an n by 1 matrix. Prove that $x^\intercal A x > 0$ if $x \neq 0$ (Hint: Write $A = P^\intercal D P$, where P is an orthogonal matrix and D a diagonal matrix possessing strictly positive diagonal entries.)

3. Compute the standard variation $\sqrt{\mathrm{Var}(\hat{R})}$ of the optimal return $\hat{R} = \sum_{i=1}^{n+1} \hat{\pi}_i R_i$. Simplify the answer for $n = 1$.

6.3 Two continuous time models

In this section we will consider n stock price processes

$$(S_i(t))_{t\geq 0},\ i = 1, ..., n$$

and one bond price process

$$B(t),\ t \geq 0$$

in continuous time. However, first it is in order to state some definitions.

A finite number of n random vectors $X_i = (X_{i1}, ..., X_{im_i}) = (X_{ik})_{k=1}^{m_i}$ in $\mathbf{R}^{m_i}$, $i = 1, ..., n$, are said to be independent if

$$E\left[\prod_{i=1}^{n} f_i(X_i)\right] = \prod_{i=1}^{n} E\left[f_i(X_i)\right]$$

for all continuous functions f_i:$\mathbf{R}^{m_i} \rightarrow \mathbf{R}$ such that $E\left[\mid f_i(X) \mid\right] < \infty$ for every $i = 1, ..., n$ (for $m_i = 1$, $i = 1, ..., n$, it can be proved that this definition is equivalent to the definition given in Chapter 3). Moreover, n stochastic processes $X_i = (X(t))_{t \in T_i}$, $i = 1, ..., n$, are said to be independent if the processes $(X(t))_{t \in T_{0i}}$ are independent for all finite subsets $T_{0i} \subseteq T_i$, $i = 1, ..., n$.

Definition 6.3.1. Let $W_i(t), t \geq 0$, $i = 1, ..., n$, be independent standard Brownian motions and set

$$W(t) = \begin{bmatrix} W_1(t) \\ : \\ W_n(t) \end{bmatrix},\ t \geq 0.$$

The process $W = (W(t))_{t \geq 0}$ is called an n-dimensional standard Brownian motion. Let σ be an invertible n by n matrix with real entries and denote by $\sigma_i = [\sigma_{i1} ... \sigma_{in}]$ the i:th row of σ and set $\mid \sigma_i \mid = \sqrt{\sigma_{i1}^2 + ... + \sigma_{in}^2}$. Furthermore, let μ be an n by 1 matrix and denote by μ_i the i:th row of μ.

(a) The model is said to be Brownian if

$$S_i(t) = S_i(0)(1 + \mu_i t + \sigma_i W(t)),\ t \geq 0,\ i = 1, ..., n$$

and

$$B(t) = B(0)(1 + rt),\ t \geq 0$$

for an appropriate $r > 0$.

(b) The model is said to be log-Brownian if

$$S_i(t) = S_i(0) e^{(\mu_i - \frac{1}{2}|\sigma_i|^2)t + \sigma_i W(t))},\ t \geq 0,\ i = 1, ..., n$$

and

$$B(t) = B(0) e^{rt},\ t \geq 0$$

for an appropriate $r > 0$.

The matrix σ in Definition 6.3.1 is called a volatility matrix.

Stock prices are hardly Brownian motions or log-Brownian motions but such a picture may be of value locally in time. Stated more explicitly, if

$$0 < T_1 < T_2 < ... < T_N$$

and the mesh size $\max_{1 \le k \le N}(T_k - T_{k-1})$ is sufficiently small it may be a good approximation to assume that the stock prices are Brownian motions or log-Brownian motions in each time interval $[T_{k-1}, T_k]$, $k = 1, ..., n$.

Suppose the return of the bond during the period from $t = T_{k-1}$ to $t = T_k$ is known at time T_{k-1} for $k = 1, ..., N$ and introduce the stock returns in the k:th period, namely

$$R_i^k = \frac{S_i(T_k) - S_i(T_{k-1})}{S_i(T_{k-1})},\ i = 1, ..., n.$$

An investor who applies the Markowitz theory in each period from $t = T_{k-1}$ to $t = T_k$ for $k = 1, ..., N$ has the possibility to estimate the model parameters in the k:th period immediately before this period starts. This is a very natural approach as the volatility of stock prices often change from one period to another. Note, however, that statistical estimates of the vector parameter μ have very big variances (cf Chapter 4, Section 2) and to evade this problem is by no means simple.

Exercises

1. Consider the Markowitz single-period problem for Brownian asset prices. Show that the Markowitz portfolio does not depend on the length T of the period.

2. Let σ and $W(t)$, $t \ge 0$, be as in Definition 6.3.1. Show that $\frac{\sigma_i}{|\sigma_i|}W(t)$, $t \ge 0$, is a standard Brownian motion for every $i = 1, ..., n$.

6.4 Problems with solutions

1. Suppose $Z = (Z_1(t), Z_2(t))_{t\geq 0}$ is a standard Brownian motion in the plane. Find

$$E\left[e^{|Z_1(t)+Z_2(t)|}\right] .$$

Solution. The process $X(t) = \frac{1}{\sqrt{2}}Z_1(t) + \frac{1}{\sqrt{2}}Z_2(t)$, $t \geq 0$, is a standard Brownian motion since $(\frac{1}{\sqrt{2}})^2 + (\frac{1}{\sqrt{2}})^2 = 1$. Hence $X(t) \in N(0,t)$ and it follows that

$$E\left[e^{|Z_1(t)+Z_2(t)|}\right] = E\left[e^{\sqrt{2}|X(t)|}\right] = E\left[e^{\sqrt{2t}|G|}\right]$$

where $G \in N(0,1)$. Thus

$$E\left[e^{|Z_1(t)+Z_2(t)|}\right] = \int_{-\infty}^{\infty} e^{\sqrt{2t}|x|-\frac{x^2}{2}}\frac{dx}{\sqrt{2\pi}} = 2\int_{-\infty}^{0} e^{-\sqrt{2t}x-\frac{x^2}{2}}\frac{dx}{\sqrt{2\pi}}$$

$$= 2e^t\int_{-\infty}^{0} e^{-\frac{1}{2}(x+\sqrt{2t})^2}\frac{dx}{\sqrt{2\pi}} = 2e^t\int_{-\infty}^{\sqrt{2t}} e^{-\frac{1}{2}x^2}\frac{dx}{\sqrt{2\pi}} = 2e^t\Phi(\sqrt{2t}).$$

2. Suppose $Z = (Z_1(t), Z_2(t))_{t\geq 0}$ is a standard Brownian motion in the plane and define $R(t) =\mid Z(t) \mid= \sqrt{Z_1^2(t)+Z_2^2(t)}$, $t \geq 0$. Find $E\left[e^{\xi R^2(t)}\right]$ if $t > 0$ and $\xi < \frac{1}{2t}$.

Solution. Suppose $t > 0$, $\xi < \frac{1}{2t}$, and $G \in N(0,1)$. Then

$$E\left[e^{\xi R^2(t)}\right] = E\left[e^{\xi Z_1^2(t)}e^{\xi Z_2^2(t)}\right] = E\left[e^{\xi Z_1^2(t)}\right]E\left[e^{\xi Z_2^2(t)}\right]$$

$$= (E\left[e^{\xi t G^2}\right])^2$$

and setting $\eta = \xi t$,

$$E\left[e^{\eta G^2}\right] = \int_{-\infty}^{\infty} e^{\eta x^2}e^{-\frac{x^2}{2}}\frac{dx}{\sqrt{2\pi}}$$

$$= \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}(1-2\eta)} \frac{dx}{\sqrt{2\pi}} = \int_{-\infty}^{\infty} e^{-\frac{y^2}{2}} \frac{dy}{\sqrt{2\pi(1-2\eta)}}$$

$$= \frac{1}{\sqrt{1-2\eta}}.$$

Hence,

$$E\left[e^{\xi R^2(t)}\right] = \frac{1}{1-2\xi t}.$$

3. Let $Z(t) = (Z_1(t), Z_2(t))$, $t \geq 0$, be a standard Brownian motion in the plane and suppose $T > 0$. Set $U = e^{2Z_1(T)}$ and $V = e^{Z_1(T)+Z_2(2T)}$. (a) Find $E[U]$, $E[V]$, $\mathrm{Var}(U)$, $\mathrm{Var}(V)$, and $\mathrm{Cov}(U,V)$. (b) Find an $a \in \mathbf{R}$ such that $\mathrm{Var}(U - aV) \leq \mathrm{Var}(U - xV)$ for every $x \in \mathbf{R}$?

Solution. (a) In the following we will use that

$$a_1 Z_1(t_1) + a_2 Z_2(t_2) \in N(0, a_1^2 t_1 + a_2^2 t_2)$$

for all $a_1, a_2 \in \mathbf{R}$ and $t_1, t_2 \geq 0$. Hence, if $G \in N(0,1)$,

$$E[U] = E\left[e^{2\sqrt{T}G}\right] = e^{2T},$$
$$E[V] = E\left[e^{\sqrt{3T}G}\right] = e^{\frac{3}{2}T},$$
$$\mathrm{Var}(U) = E[U^2] - (E[U])^2 = E\left[e^{4\sqrt{T}G}\right] - e^{4T} = e^{8T} - e^{4T},$$
$$\mathrm{Var}(V) = E[V^2] - (E[V])^2 = E\left[e^{2\sqrt{3T}G}\right] - e^{3T} = e^{6T} - e^{3T},$$
$$\mathrm{Cov}(U,V) = E[UV] - E[U]\,E[V] = E\left[e^{\sqrt{11T}G}\right] - e^{2T}e^{\frac{3}{2}T} = e^{\frac{11}{2}T} - e^{\frac{7}{2}T}.$$

(b) Set $U_0 = U - E[U\ ]$ and $V_0 = V - E[V]$. We have

$$f(x) =_{def} \mathrm{Var}(U - xV) = E\left[(U_0 - xV_0)^2\right]$$

$$= E\left[U_0{}^2\right] - 2xE[U_0V_0] + x^2 E\left[V_0{}^2\right]$$

$$= (x\sqrt{E[V_0{}^2]} - \frac{E[U_0V_0]}{\sqrt{E[V_0{}^2]}})^2 + E\left[U_0{}^2\right] - (\frac{E[U_0V_0]}{\sqrt{E[V_0{}^2]}})^2.$$

Hence

$$\min f = f(a)$$

where

$$a = \frac{\mathrm{Cov}(U,V)}{\mathrm{Var}(V)} = \frac{e^{\frac{11}{2}T} - e^{\frac{7}{2}T}}{e^{6T} - e^{3T}}$$

$$= \frac{e^{\frac{5}{2}T} - e^{\frac{1}{2}T}}{e^{3T} - 1} = \frac{e^{\frac{1}{2}T}(e^{T}+1)}{e^{2T}+e^{T}+1}.$$

4. Let $T > 0$ and consider two stock price processes

$$\begin{cases} S_1(t) = S_1(0)e^{\alpha_1 t + \sigma_1 W_1(t)}, \ 0 \le t \le T \\ S_2(t) = S_2(0)e^{\alpha_2 t + \sigma_2 W_2(t)}, \ 0 \le t \le T \end{cases}$$

governed by a bivariate geometric Brownian motion with correlation parameter $\rho \in \,]-1,1[\,$. A portfolio is long 1000 shares of the first stock and short $\frac{1000 S_1(0)}{S_2(0)}$ shares of the second stock. Consequently, the corresponding portfolio $\mathcal{A}$ is of value zero at time zero, that is $V_{\mathcal{A}}(0) = 0$. Find $P\left[V_{\mathcal{A}}(T) > 0\right]$, $E\left[V_{\mathcal{A}}(T)\right]$, and $E\left[(V_{\mathcal{A}}(T))^2\right]$.

Solution. We have

$$V_{\mathcal{A}}(T) = K(e^{\alpha_1 T + \sigma_1 W_1(T)} - e^{\alpha_2 T + \sigma_2 W_2(T)})$$

where $K = 1000 S_1(0)$. Hence

$$P\left[V_{\mathcal{A}}(T) > 0\right] = P\left[e^{\alpha_1 T + \sigma_1 W_1(T)} > e^{\alpha_2 T + \sigma_2 W_2(T)}\right]$$

$$= P\left[\sigma_1 W_1(T) - \sigma_2 W_2(T) > (\alpha_2 - \alpha_1)T\right].$$

Set $X_{\pm} = \sigma_1 W_1(T) \pm \sigma_2 W_2(T) \in N(0, \sigma_{\pm}^2 T)$, where

$$\sigma_{\pm}^2 T =_{def} E\left[(\sigma_1 W_1(T) \pm \sigma_2 W_2(T))^2\right] = (\sigma_1^2 \pm 2\rho\sigma_1\sigma_2 + \sigma_2^2)T.$$

Now

$$P\left[V_{\mathcal{A}}(T) > 0\right] = \Phi\left(\frac{(\alpha_1 - \alpha_2)\sqrt{T}}{\sqrt{\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2}}\right).$$

Moreover, if $G \in N(0,1)$,

$$E\left[e^{\xi G}\right] = e^{\frac{\xi^2}{2}}, \ \xi \in \mathbf{R}$$

and it follows that

$$E\left[V_{\mathcal{A}}(T)\right] = K(e^{(\alpha_1+\frac{1}{2}\sigma_1^2)T} - e^{(\alpha_2+\frac{1}{2}\sigma_2^2)T})$$

and

$$E\left[(V_{\mathcal{A}}(T))^2\right]$$
$$= K^2 E\left[e^{2\alpha_1 T+2\sigma_1 W_1(T)} - 2e^{(\alpha_1+\alpha_2)T+\sigma_1 W_1(T)+\sigma_2 W_2(T)} + e^{2\alpha_2 T+2\sigma_2 W_2(T)}\right]$$
$$= K^2(e^{2(\alpha_1+\sigma_1^2)T} - 2e^{(\alpha_1+\alpha_2+\frac{1}{2}\sigma_1^2+\rho\sigma_1\sigma_2+\frac{1}{2}\sigma_2^2)T} + e^{2(\alpha_2+\sigma_2^2)T}).$$

5. Let W be a standard Brownian motion and set $U = W^2(1)$ and $V = W(1)W(2)+W(3)$. Find (a) $E[U]$ (b) $E[V]$ (c) $E[U^2]$ (d) $E[V^2]$ (e) $E[UV]$ (f) $\operatorname{Cov}(U,V)$ and (g) $\operatorname{Cor}(U,V)$.

Solution. Set $X = W(1)$, $Y = W(2) - W(1)$, and $Z = W(3) - W(2)$. Then X, Y, and Z are independent, $X, Y, Z \in N(0,1)$, and

$$U = X^2$$

and

$$V = X^2 + XY + X + Y + Z.$$

(a)
$E[U] = E[X^2] = 1$

(b)
$E[V] = E[X^2] + E[X]E[Y] + E[X] + E[Y] + E[Z] = 1$

(c)
$E[U^2] = E[X^4] = 3$

(d) $E[V^2] = E\left[\{X(X+Y) + (X+Y+Z)\}^2\right] = E[X^2(X^2+2XY+Y^2)] + 2E[(X^2+XY)(X+Y+Z)] + E[(X+Y+Z)^2] = (E[X^4] + E[X^2Y^2]) + 0 + \operatorname{Var}(X+Y+Z) = 3+1+3 = 7$

Alternative solution:

$E[V^2] = E[X^4]+E[X^2]E[Y^2]+E[X^2]+E[Y^2]+E[Z^2]+2E[X^3]E[Y]+ 2E[X^3]+2E[X^2]E[Y]+2E[X^2]E[Z]+E[X^2]E[Y]+E[X]E[Y^2]+ E[X]E[Y]E[Z]+E[X]E[Y]+E[X]E[Z]+E[Y]E[Z] = 3+1+1+ 1+1+0+0+0+0+0+0+0+0+0+0 = 7$

(e)
$E[UV] = E[X^4]+E[X^3]E[Y]+E[X^3]+E[X^2]E[Y]+E[X^2]E[Z] = 3+0+0+0+0 = 3$

(f)
$\text{Cov}(U,V) = 2$

(g)
$\text{Cor}(U,V) = \frac{2}{\sqrt{2}\sqrt{6}} = \frac{1}{\sqrt{3}}$

CHAPTER 7

Dividend-Paying Stocks

Introduction

Suppose a stock has the price process $S = (S(t))_{t \geq 0}$ and pays the dividend $D > 0$ at time t^*. We have the convention that $S(t^*-)$ and $S(t^*+)$ exist, $S(t^*) = S(t^*+)$ and

$$S(t^*-) - S(t^*) = D.$$

In particular, the process S is no longer continuous and the Black-Scholes option pricing must be modified, which is the subject of this chapter. Clearly, dividends are something very important and quantitative analysts spend lots of time to price options on dividend-paying stocks.

7.1 A Seek for Portfolios with a Geometric Brownian Motion Dynamics

Consider a European derivative on S with the payoff $g(S(T))$ at the termination date T and, in addition, suppose $g \in \mathcal{P}$. First suppose there is only one dividend during the life time of the option and this occurs at time t^*. Moreover, assume S is a geometric Brownian motion restricted to the time interval $[t^*, T]$ and denote the corresponding volatility by σ. Let $t < t^*$ be the present date. We then try to find a portfolio $\mathcal{A}$ containing only the stock and bond such that process

$$S^*(\lambda) = \begin{cases} V_{\mathcal{A}}(\lambda), \ t \leq \lambda < t^* \\ S(\lambda), \ t^* \leq \lambda \leq T \end{cases}$$

is governed by a geometric Brownian motion with volatility σ. In particular,

$$V_{\mathcal{A}}(t^*-) = S(t^*).$$

By selling the portfolio $\mathcal{A}$ immediately before time t^* and using the payment obtained to buy the stock at time t^* when the dividend is detached, the process $(S^*(\lambda))_{\, t\leq\lambda\leq T}$ can be viewed as the price process of a traded security. Therefore, as $g(S(T)) = g(S^*(T))$, it is natural to define

$$\Pi_{g(S(T))}(t) = e^{-r\tau} E\left[g(s^* e^{(r-\frac{\sigma^2}{2})\tau+\sigma\sqrt{\tau}G}) \right]_{|s^*=S^*(t)}$$

Here, as usual, $\tau = T - t$.

First assume the dividend paid equals

$$D = \delta S(t^*-)$$

where $\delta \in]0,1[$ is a real number known at time t. To determine the portfolio $\mathcal{A}$ suppose S is governed by a geometric Brownian motion with volatility σ before time t^*. If $\mathcal{A}$ as a portfolio containing $(1-\delta)$ units of the stock, it is natural to assume that S^* is a geometric Brownian motion with volatility σ and

$$\Pi_{g(S(T))}(t) = e^{-r\tau} E\left[g((1-\delta) s e^{(r-\frac{\sigma^2}{2})\tau+\sigma\sqrt{\tau}G} \right]_{|s=S(t)},\ t < t^*.$$

Consequently, a dividend in terms of fractions of the stock price is very simple to handle.

Next suppose the dividend paid at time t^* is a fixed amount D in Swedish crowns. To handle this case we assume that the process

$$S(\lambda) - De^{-r(t^*-\lambda)},\ t \leq \lambda \leq t^*$$

is a geometric Brownian motion with volatility σ and let $\mathcal{A}$ be a portfolio consisting of a stock and a short position in the bond corresponding to $D/B(t^*)$ units. Now

$$s^* = S(t) - De^{-r(t^*-t)}$$

and assuming that S^* is a geometric Brownian motion with volatility σ,

$$\Pi_{g(S(T))}(t) = e^{-r\tau} E\left[g((s - De^{-r(t^*-t)}) e^{(r-\frac{\sigma^2}{2})\tau+\sigma\sqrt{\tau}G} \right]_{|s=S(t)}$$

To find an approximate option price based on the binomial approximation at time t, define $h = \tau/N$,

$$t_n = t + nh,\ n = 0, 1, ..., N$$

and

$$v_j^N = g(s^* e^{(N-2j)\sigma\sqrt{h}}),\ j = 0, 1, ..., N\,.$$

Next at the times $t_n, n = N-1, N-2, ..., 1, 0$, we compute the corresponding option prices

$$v_j^n = e^{-rh}(q_u v_j^{n+1} + q_d v_{j+1}^{n+1})$$

for $j = 0, 1, ..., n$, where

$$q_u = 1 - q_d = \frac{e^{rh} - e^{-\sigma\sqrt{h}}}{e^{\sigma\sqrt{h}} - e^{-\sigma\sqrt{h}}}.$$

The quantity v_0^0 approximates $\Pi_{g(S(T))}(t)$.

The price of the corresponding American option has not been defined in this text but intuition leads us to the following algorithm. First let $g_j^n =$for $j = 0, 1, ..., n$ and, as above, let $v_j^N = g(s_* e^{(N-2j)\sigma\sqrt{h}}),\ j = 0, 1, ..., N\,.$ Next at the times $t_n, n = N-1, N-2, ..., 1, 0$, we compute the corresponding option prices

$$v_j^n = \max(g_j^n, e^{-rh}(pv_j^{n+1} + qv_{j+1}^{n+1}))$$

for $j = 0, 1, ..., n$. Finally, the quantity v_0^0 approximates the American option price at time t.

Finally, in this chapter we will discuss two qualitative properties of dividend-paying American calls and puts.

Exercises

1. A forward starting European put on S has the payoff

$$\max(0, S(T_0) - S(T))$$

   at the termination date T. Suppose $T_0 < t^* < T$, $0 < \delta < 1$, and that the stock pays the dividend $\delta S(t^*-)$ at time t^*. Find the call price for $t < t^*$.

2. A European derivative on S has the payoff $\max(S(T), K)$ at the termination time T. The stock pays the dividend D at time t^*, where D is a known amount at time $t < t^*$. Find the price of the derivative at time t.

3. A European derivative on S has the payoff $g(S(T))$ at maturity T, where $g \in \mathcal{P}$.

   (a) Suppose there are times $t < t_1 < ... < t_n < T$ and, at each t_k, the dividend paid is $\delta S(t_k-)$. Moreover, let $\mathcal{A}$ be a dynamic portfolio with exactly $(1-\delta)^k$ units of the stock in the interval $[t_{n-k}, t_{n-k+1}[$, $k = 0, 1, ..., n$, where $t_0 = t$ and $t_{n+1} = T$. Suppose the portfolio process $(V_{\mathcal{A}}(\lambda))_{t \leq \lambda \leq T}$ is a geometric Brownian motion with volatility σ. Motivate the following definition, namely

$$\Pi_{g(S(T))}(t) = e^{-r\tau} E\left[g((1-\delta)^n s e^{(r-\frac{\sigma^2}{2})\tau + \sigma\sqrt{\tau}G})\right]_{|s=S(t)}.$$

   (b) Suppose $\delta > 0$ and that the dividend paid in the interval $[t, t+dt[$ equals $\delta S(t)dt$ for each t. Suppose the stock price is governed by a geometric Brownian motion with volatility σ. Motivate the following definition, namely

$$v(t,s) = e^{-r\tau} E\left[g(s e^{(r-\delta-\frac{\sigma^2}{2})\tau + \sigma\sqrt{\tau}G})\right]$$

4. Suppose the dividend paid at time t^* is a fixed amount D in Swedish crowns and assume, in contrast to the assumptions above, that the process

$$S^*(\lambda) = \begin{cases} S(\lambda), \ t \leq \lambda < t^* \\ S(\lambda) + De^{r(\lambda - t^*)}, \ t^* \leq \lambda \leq T \end{cases}$$

   is a geometric Brownian motion with volatility σ. Find the price of a call on S with strike K and termination date T.

7.2 A Property of the American Call when the underlying pays a dividend

Consider an American call on S with strike K and time of maturity T, let $t < t_1 < t^* < T$ and suppose the stock pays the dividend D in Swedish crowns at time t^*. Only assuming the dominance principle in Chapter 1, we claim that it is not optimal to exercise the option in the interval $[t, t_1]$. Indeed, if $t_1 < t_2 < t^*$,

$$C(t_2, S(t_2), K, T) \geq S(t_2) - K$$

or

$$C(t_2, S(t_2), K, T) \geq S(t_2) - \frac{K}{B(t_2)} B(t_2)$$

and an application of the dominance principle yields

$$C(\lambda, S(\lambda), K, T) \geq S(\lambda) - \frac{K}{B(t_2)} B(\lambda) > S(\lambda) - K \text{ if } \lambda \leq t_1.$$

However, in some cases it is optimal to exercise the call immediately before the dividend is detached.

Since it is not optimal to exercise the option in the interval $[t^*, T[$,

$$C(t^*, S(t^*), K, T) = c(t^*, S(t^*), K, T).$$

The exercise value at time t^*- equals

$$S(t^*-) - K$$

and, accordingly from this,

$$C(t^*-, S(t^*-), K, T) = \max(S(t^*-) - K, c(t^*, S(t^*), K, T))$$

or, since $S(t^*-) = S(t^*) + D$,

$$C(t^*-, S(t^*-), K, T) = \max(S(t^*) - (K - D), c(t^*, S(t^*), K, T)).$$

Now we assume the stock price is governed by a geometric Brownian motion with volatility σ in the time interval $[t^*, T]$ and recall that

$$\frac{\partial c}{\partial s} = \Phi(d_1) < 1.$$

Therefore there is at most one positive number s_C such that

$$S(t^*) - (K - D) > c(t^*, S(t^*), K, T)) \text{ if } S(t^*) > s_C$$

and

$$S(t^*) - (K - D) < c(t^*, S(t^*), K, T)) \text{ if } S(t^*) < s_C.$$

If such a number s_C exists and

$$S(t^*-) > s_C + D$$

it is optimal to exercise the option at time $t^* -$. On the other hand, if $\tau^* = T - t^*$ and

$$D \le K(1 - e^{-r\tau^*})$$

it is not optimal to exercise the option at time t^*- regardless of the value on $S(t^*-)$. To see this, we observe that

$$S(t^*) - (K - D) \le S(t^*) - Ke^{-r\tau^*}$$

and

$$c(t^*, S(t^*), K, T)) = e^{-r\tau^*} E\left[\max(0, se^{(r-\frac{\sigma^2}{2})\tau^* + \sigma\sqrt{\tau^*}G} - K)\right]_{|s=S(t^*)}$$

$$= E\left[\max(0, se^{-\frac{\sigma^2}{2}\tau^* + \sigma\sqrt{\tau^*}G} - e^{-r\tau^*}K)\right]_{|s=S(t^*)}$$

$$= \int_{-\infty}^{\infty} \max(0, S(t^*)e^{-\frac{\sigma^2}{2}\tau^* + \sigma\sqrt{\tau^*}x} - e^{-r\tau^*}K)\varphi(x)dx$$

$$> \int_{-\infty}^{\infty} (S(t^*)e^{-\frac{\sigma^2}{2}\tau^* + \sigma\sqrt{\tau^*}x} - e^{-r\tau^*}K)\varphi(x)dx$$

$$= S(t^*) - Ke^{-r\tau^*}.$$

Thus

$$c(t^*, S(t^*), K; T)) > S(t^*) - (K - D).$$

7.3 A Property of the American Put when the underlying pays a dividend

Consider an American put on S with strike K and time of maturity T and suppose the stock pays the dividend D in Swedish crowns at time t^*, where $t^* < T$. The dividend D is known at time t_0, where $t_0 < t^*$. If

$$D \ge K(e^{r(t^* - t_0)} - 1)$$

we claim that it is not optimal to exercise the put in the time interval $]t_0, t^*[$. It is enough to prove that $P(t, S(t), K, T) > K - S(t)$ if $t \in \,]t_0, t^*[$. In fact, if $t \in \,]t_0, t^*[$ and $P(t, S(t), K, T) = K - S(t)$, let $\mathcal{A}$ be a portfolio consisting of long positions in the American put and the stock, and a short position in the bond corresponding to $-K/B(t)$ units. Then

$$V_{\mathcal{A}}(t) = (K - S(t)) + S(t) - K = 0.$$

However,

$$\begin{aligned} V_{\mathcal{A}}(t^*) &= D + P(t^*, S(t^*), K, T) + S(t^*) - \frac{K}{B(t)} B(t^*) \\ &\geq D + (K - S(t^*)) + S(t^*) - \frac{K}{B(t)} B(t^*) \\ &= D + K - \frac{K}{B(t)} B(t^*) \\ &= D + K(1 - e^{r(t^* - t)}) > D + K(1 - e^{r(t^* - t_0)}) \geq 0 \end{aligned}$$

which contradicts (a slightly more general version of) the dominance principle (than we stated in Chapter 1). Hence $P(t, S(t), K; T) > K - S(t)$ and it is not optimal to exercise the American put in the time interval $]t_0, t^*[$.

Problems with solutions

1. (Black-Scholes model) Suppose $0 < T_0 < t^* < T$ and $0 < \delta < 1$ and consider a European-style derivative with the payoff

$$Y = \mid S(T) - S(T_0) \mid$$

at time of maturity T. Find $\Pi_Y(0)$ if the stock pays the dividend $\delta S(T_0)$ at time t^*.

Solution. First note that

$$Y = 2(S(T) - S(T_0))^+ - S(T) + S(T_0).$$

If $s_0 = S(T_0)$ and

$$g(x) = 2(x - s_0)^+ - x + s_0$$

then

$$\begin{aligned}\Pi_Y(T_0) &= \Pi_{g(S(T))}(T_0)\\ &= e^{-r(T-T_0)}E\left[g((s_0 - \delta s_0 e^{-r(t^*-T_0)})e^{(r-\frac{\sigma^2}{2})(T-T_0)+\sigma\sqrt{T-T_0}G})\right]\\ &= e^{-r(T-T_0)}E\left[g((s_0(1 - \delta e^{-r(t^*-T_0)})e^{(r-\frac{\sigma^2}{2})(T-T_0)+\sigma\sqrt{T-T_0}G})\right]\end{aligned}$$

where $G \in N(0,1)$. Hence

$$\Pi_Y(T_0) = 2c(T_0, s_0(1-\delta e^{-r(t^*-T_0)}), s_0, T) - s_0(1-\delta e^{-r(t^*-T_0)}) + s_0 e^{-r(T-T_0)}$$

and we get

$$\Pi_Y(T_0) = S(T_0)\left\{(1-\delta e^{-r(t^*-T_0)})A - e^{-r(T-T_0)}B - 1 + \delta e^{-r(t^*-T_0)} + e^{-r(T-T_0)}\right\}$$

where

$$A = 2\Phi(\frac{\ln(1-\delta e^{-r(t^*-T_0)}) + (r+\frac{\sigma^2}{2})(T-T_0)}{\sigma\sqrt{T-T_0}})$$

and

$$B = 2\Phi(\frac{\ln(1-\delta e^{-r(t^*-T_0)}) + (r-\frac{\sigma^2}{2})(T-T_0)}{\sigma\sqrt{T-T_0}}).$$

Since A and B are independent of $S(T_0)$ we conclude that

$$\Pi_Y(0) = S(0)\left\{(1-\delta e^{-r(t^*-T_0)})A - e^{-r(T-T_0)}B - 1 + \delta e^{-r(t^*-T_0)} + e^{-r(T-T_0)}\right\}.$$

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