Solutions Exercises 1)

To solve the exercises, the following tools are required:

- Dominance principle (Ref. [1], pag. 22)
- Ref. [2], Theorem 1.1.1

Ref. [2], Sec. 1.1, Ex. 1 (Butterfly Spread Strategy)

Suppose $\Delta K > 0$. A butterfly spread on call options pays at the maturity T the amount

$$\max(0, S(T) - K + \Delta K) - 2\max(0, S(T) - K) + \max(0, S(T) - K - \Delta K).$$

Show that the value of this option is non-negative at any point of time.

Solution

Consider the following assets:

- $\mathcal{U}_1 \equiv$ European Call with strike $K - \Delta K$ and maturity T
- $\mathcal{U}_2 \equiv$ European Call with strike K and maturity T
- $\mathcal{U}_3 \equiv$ European Call with strike $K + \Delta K$ and maturity T

and set up the portfolio position

$$\mathcal{A} = (1, -2, 1).$$

This means that the investor buys one share of $\mathcal{U}_1$ and $\mathcal{U}_3$ and sell 2 shares of the call $\mathcal{U}_2$. The value at maturity of this portfolio is

$$V_{\mathcal{A}}(T) = (S(T) - (K - \Delta K))_+ - 2(S(T) - K)_+ + (S(T) - (K + \Delta K))_+$$

where $(x)_+ = \max(0, x)$. Hence

- If $S(T) \leq K - \Delta K$, $V_{\mathcal{A}}(T) = 0 + 0 + 0 = 0$
- If $K - \Delta K < S(T) \leq K$, $V_{\mathcal{A}}(T) = S(T) - (K - \Delta K) + 0 + 0 = S(T) - K + \Delta K > 0$

- If $K < S(T) < K + \Delta K$, $V_{\mathcal{A}}(T) = S(T) - (K - \Delta K) - 2(S(T) - K) + 0 = -S(T) + K + \Delta K > 0$
- If $S(T) \geq K + \Delta K$, $V_{\mathcal{A}}(T) = S(T) - (K - \Delta K) - 2(S(T) - K) + S(T) - (K + \Delta K) = 0$

It follows that this portfolio is lucrative if and only if $S(T) \in (K - \Delta K, K + \Delta K)$. Moreover $V_{\mathcal{A}}(T) \geq 0$, and so $V_{\mathcal{A}}(t) \geq 0$, for all $t \in [0, T]$, by the dominance principle.

Ref. [2], Sec. 1.1, Ex. 3

The price of a contract at time t is N units of currency and it pays at the maturity date $T > t$ the amount $N + \alpha N(S(T) - K)_+$. Show that

$$\alpha = \frac{1 - e^{-r(T-t)}}{C(t, S(t), K, T)} \tag{1}$$

if $C(t, S(t), K, T) > 0$ and $N \neq 0$.

Solution

Consider the following assets:

- $\mathcal{U}_1 \equiv$ Contract
- $\mathcal{U}_2 \equiv$ Risk-free asset with initial value 1
- $\mathcal{U}_3 \equiv$ European Call with strike K and maturity T

Consider the two portfolios

$$\mathcal{A} = (0, Ne^{-rT}, \alpha N), \quad \mathcal{B} = (1, 0, 0).$$

We want to show that the portfolio $\mathcal{A}$ on the call and the risk-free asset replicates the value of the contract. To this purpose we observe that

$$V_{\mathcal{A}}(t) = Ne^{-r(T-t)} + \alpha N C(t, S(t), K, T), \tag{2}$$

while $V_{\mathcal{B}}(t) = N$ by assumption. Letting $t = T$ in (2) we obtain

$$V_{\mathcal{A}}(T) = N + \alpha N(S(T) - K)_+ = V_{\mathcal{B}}(T),$$

hence, by the dominance principle, $V_{\mathcal{A}}(t) = V_{\mathcal{B}}(t)$. Using (2) and $V_{\mathcal{B}}(t) = N$ we obtain

$$Ne^{-r(T-t)} + \alpha N C(t, S(t), K, T) = N$$

and since $C(t, S(t), K, T) > 0$ and $N \neq 0$, we obtain (1).

Ref. [2], Sec. 1.1, Ex. 4

Suppose H is the Heaviside function, i.e., $H(x) = 0$, if $x < 0$ and $H(x) = 1$ if $x \geq 0$. Moreover suppose that K and ΔK are positive and $K - \Delta K > 0$ A digital call option with cash settlement has the pay-off function $Y_0 = H(S(T) - K)$ at the time of maturity T and a digital call option with physical settlement has the pay-off function $Y_1 = S(T)H(S(T) - K)$ at time of maturity T. Digital options are also known as binary or bet options. Below we assume that the contracts are of European type.

(a) Show that

$$C(t, S(t), K, T) - C(t, S(t), K + \Delta K, T) \leq \Delta K \, \Pi_{Y_0}(t) \leq C(t, S(t), K - \Delta K, T) - C(t, S(t), K, T) \tag{3}$$

and conclude that

$$\Pi_{Y_0}(t) = -\frac{\partial C}{\partial K}(t, S(t), K, T)$$

if the derivative exists.

(b) Show that

$$\Pi_{Y_1}(t) = C(t, S(t), K, T) + K\Pi_{Y_0}(t).$$

Solution

(a) Consider the following assets:

- $\mathcal{U}_1 \equiv$ European Call with strike K and maturity T
- $\mathcal{U}_2 \equiv$ European Call with strike $K + \Delta K$ and maturity T
- $\mathcal{U}_3 \equiv$ European Call with strike $K - \Delta K$ and maturity T
- $\mathcal{U}_4 \equiv$ Digital option with pay-off Y_0 at maturity T

and the following portfolios positions:

$$\mathcal{A} = (1, -1, 0, 0), \quad \mathcal{B} = (0, 0, 0, \Delta K), \quad \mathcal{C} = (-1, 0, 1, 0).$$

We have

$$V_{\mathcal{A}}(T) = (S(T) - K)_+ - (S(T) - (K + \Delta K))_+ = \begin{cases} 0, & \text{if } S(T) \leq K \\ S(T) - K, & \text{if } K < S(T) \leq K + \Delta K \\ \Delta K, & \text{if } S(T) > K + \Delta K \end{cases}$$

$$V_{\mathcal{B}}(T) = \Delta K H(S(T) - K) = \begin{cases} 0, & \text{if } S(T) \leq K \\ \Delta K, & \text{if } S(T) > K \end{cases}$$

$$V_{\mathcal{C}}(T) = -(S(T)-K)_+ + (S(T)-(K-\Delta K))_+ = \begin{cases} 0, & \text{if } S(T) \leq K - \Delta K \\ S(T) - K + \Delta K, & \text{if } K - \Delta K < S(T) \leq K \\ \Delta K, & \text{if } S(T) > K \end{cases}$$

It follow that

$$V_{\mathcal{A}}(T) \leq V_{\mathcal{B}}(T) \leq V_{\mathcal{C}}(T)$$

(draw pictures of the portfolios values as functions of $S(T)$). Hence, by the dominance principle,

$$V_{\mathcal{A}}(t) \leq V_{\mathcal{B}}(t) \leq V_{\mathcal{C}}(t)$$

which is equivalent to (3). Dividing by $-\Delta K$ we obtain

$$\frac{C(t, S(t), K, T) - C(t, S(t), K - \Delta K, T)}{\Delta K} \leq -\Pi_{Y_0}(t) \leq \frac{C(t, S(t), K + \Delta K, T) - C(t, S(t), K, T)}{\Delta K}.$$

Taking the limit $\Delta K \to 0$ and assuming that c is differentiable in K, the first and last member of the previous inequality converge to $\partial C/\partial K$. Hence $\Pi_{Y_0}(t) = -\partial C/\partial K$ as claimed.

(b) Consider the assets

- $\mathcal{U}_1 \equiv$ Digital option with pay-off Y_1
- $\mathcal{U}_2 \equiv$ European Call with strike K and maturity T
- $\mathcal{U}_3 \equiv$ Digital option with pay-off Y_0

and the portfolios

$$\mathcal{A} = (1, 0, 0), \quad \mathcal{B} = (0, 1, K)$$

Then $V_{\mathcal{A}}(t) = S(T)H(S(T) - K)$ and

$$V_{\mathcal{B}}(t) = (S(T) - K)_+ + K\,H(S(T) - K) = \begin{cases} 0, & \text{if } S(T) < K \\ S(T), & \text{if } S(T) \geq K \end{cases}$$

Hence $V_{\mathcal{A}}(T) = V_{\mathcal{B}}(T)$ and by the dominance principle, $V_{\mathcal{A}}(t) = V_{\mathcal{B}}(t)$, which is the claim.

1 Ref. [1], Ex. 1.7

Assume that the dominance principle holds. Consider a European derivative $\mathcal{U}$ with maturity time T and pay-off Y given by $Y = \min[(S(T) - K_1)_+, (K_2 - S(T))_+]$, where $K_2 > K_1$ and $(x)_+ = max(0, x)$. Find a constant portfolio consisting of European calls and puts expiring at time T whose value at any time $t < T$ equals the value of $\mathcal{U}$ (i.e., which replicates the value of $\mathcal{U}$).

Solution

To solve this exercise it is useful to begin by drawing the pay-off as a function of $S(T)$. Since

$$Y = \min\left[(S(T) - K_1)_+, (K_2 - S(T))_+\right],$$

where $K_2 > K_1$ and $(x)_+ = \max(0, x)$, then we first draw the functions $S(T) \to (S(T)-K_1)_+$ and $S(T) \to (K_2 - S(T))_+$ and then we take their minimum.

Now let $\mathcal{A}$ be a portfolio that consists of one share of the derivative and let $V_{\mathcal{A}}(t)$ be its value at time $t \in [0, T]$. The exercise asks to derive a portfolio $\mathcal{B}$ consisting of European calls and puts which replicates the value of $\mathcal{A}$, i.e., such that $V_{\mathcal{A}}(t) = V_{\mathcal{B}}(t)$, for all $t \in [0, T]$. By the dominance principle (in particular by (b) of Theorem 1.1.1 in Borell's notes) it is enough to find the portfolio $\mathcal{B}$ in such a way that $V_{\mathcal{A}}(T) = V_{\mathcal{B}}(T)$. By definition, $V_{\mathcal{A}}(T) = Y$, since the value of a derivative at the expiration date always equals the pay-off. So we we have to find a combination of pay-off functions of puts and calls such that their sum equal Y. By using the graph constructed before it is easy to see that

$$Y = (S(T) - K_1)_+ - 2(S(T) - \frac{K_1 + K_2}{2})_+ + (S(T) - K_2)_+$$

Even without making any drawing, the previous identity can be verified by direct calculation. Hence, letting

$$\begin{aligned}
\mathcal{U}_1 &= \text{European call with strike } K_1 \\
\mathcal{U}_2 &= \text{European call with strike } (K_1 + K_2)/2 \\
\mathcal{U}_3 &= \text{European call with strike } K_2
\end{aligned}$$

where for all calls the expiration date is T, we take

$$\mathcal{B} = (1, -2, 1).$$

This concludes the solution of the exercise.