# Solutions to selected exercises (week 6)

## Exercise 5.6, Ref. [1]

Show that when $X, Y$ are independent random variables, then the only events which are resolved by both variables are $\emptyset$ and $\Omega$. Show that two deterministic constants are always independent. Finally assume $Y = g(X)$ and show that in this case the two random variables are independent if and only if $Y$ is a deterministic constant.

### Solution

Let $A$ be an event that is resolved by both variables $X, Y$. This means that there exist $I, J \subseteq \mathbb{R}$ such that $A = \{X \in I\} = \{Y \in J\}$. Hence, using the independence of $X, Y$,

$$\mathbb{P}(A) = \mathbb{P}(A \cap A) = \mathbb{P}(X \in I, Y \in J) = \mathbb{P}(X \in I)\mathbb{P}(Y \in J) = \mathbb{P}(A)\mathbb{P}(A) = \mathbb{P}(A)^2.$$

Therefore $\mathbb{P}(A) = 0$ or $\mathbb{P}(A) = 1$. In a finite probability space this implies $A = \emptyset$ or $A = \Omega$, respectively.

Now let $a, b$ be two deterministic constants. Note that, for all $I \subset \mathbb{R}$,

$$\mathbb{P}(a \in I) = \begin{cases} 1 & \text{if } a \in I \\ 0 & \text{otherwise} \end{cases}$$

and similarly for $b$. Hence

$$\mathbb{P}(a \in I, b \in J) = \begin{cases} 1 & \text{if } a \in I \text{ and } b \in J \\ 0 & \text{otherwise} \end{cases} = \mathbb{P}(a \in I)\mathbb{P}(b \in J).$$

Finally we show that $X$ and $Y = g(X)$ are independent if and only if $Y$ is a deterministic constant. For the "if" part we use that

$$\mathbb{P}(a \in I, X \in J) = \begin{cases} \mathbb{P}(X \in J) & \text{if } a \in I \\ 0 & \text{otherwise} \end{cases} = \mathbb{P}(a \in I)\mathbb{P}(X \in J).$$

For the "only if" part, let $z \in \mathbb{R}$ and $I = \{g(X) \leq z\} = \{X \in g^{-1}(-\infty, z]\}$. Then, using the independence of $X$ and $Y = g(X)$,

$$\begin{aligned} \mathbb{P}(g(X) \leq z) &= \mathbb{P}(g(X) \leq z, g(X) \leq z) = \mathbb{P}(X \in g^{-1}(-\infty, z], g(X) \leq z) \\ &= \mathbb{P}(X \in g^{-1}(-\infty, z])\mathbb{P}(g(X) \leq z) = \mathbb{P}(g(X) \leq z)\mathbb{P}(g(X) \leq z). \end{aligned}$$

Hence $\mathbb{P}(Y \leq z)$ is either 0 or 1, which implies that $Y$ is a deterministic constant.

# Exercise 5.7, Ref. [1]

The exercise asks to prove the following:

Let $X_1, X_2$ be independent random variables, $g : \mathbb{R} \to \mathbb{R}$, and $f : \mathbb{R} \to \mathbb{R}$. Then the random variables

$$Y = g(X_1), \quad Z = f(X_2)$$

are independent. Moreover

$$\mathrm{Var}[X_1 + X_2] = \mathrm{Var}[X_1] + \mathrm{Var}[X_2]$$

## Solution

Given $I, J \subseteq \mathbb{R}$ we have $\{Y \in I\} = \{X_1 \in \{g \in I\}\}$ and $\{Z \in J\} = \{X_2 \in \{f \in J\}\}$. Hence, using the independence of $X_1, X_2$,

$$\begin{aligned}\mathbb{P}(Y \in I, Z \in J) &= \mathbb{P}(X_1 \in \{g \in I\}, X_2 \in \{f \in J\}) \\ &= \mathbb{P}(X_1 \in \{g \in I\})\mathbb{P}(X_2 \in \{f \in J\}) = \mathbb{P}(Y \in I)\mathbb{P}(Z \in J).\end{aligned}$$

As to the second statement we write

$$\mathrm{Var}[X_1+X_2] = \mathbb{E}[(X_1+X_2)^2]-\mathbb{E}[(X_1+X_2)]^2 = \mathrm{Var}[X_1]+\mathrm{Var}[X_2]+2(\mathbb{E}[X_1X_2]-\mathbb{E}[X_1]\mathbb{E}[X_2]),$$

hence the claim follows if we show that $\mathbb{E}[X_1X_2] = \mathbb{E}[X_1]\mathbb{E}[X_2]$, i.e., the two random variables are uncorrelated. This is shown in Exercise 5.8 below.

# Exercise 5.8, Ref. [6]

Let $(\Omega, \mathbb{P})$ be a finite probability space and $X, Y : \Omega \to \mathbb{R}$ be two random variables. Prove that $X, Y$ independent $\Rightarrow$ $X, Y$ uncorrelated. Show with a counterexample that the opposite implication is not true. Prove the inequality

$$-\sqrt{\mathrm{Var}[X]\mathrm{Var}[Y]} \leq \mathrm{Cov}(X, Y) \leq \sqrt{\mathrm{Var}[X]\mathrm{Var}[Y]}. \tag{1}$$

Now assume that $X, Y$ have positive variance (i.e., they are not deterministic constants[1]). Show that the left (resp. right) inequality becomes an equality if and only if there exists a negative (resp. positive) constant $a_0$ and a real constant $b_0$ such that $Y = a_0X + b_0$.

## Solution

The statement holds for random variables on general probability spaces, but here we are only concerned with finite probability spaces. In particular, $X$ can only take a finite number of values $x_1, \ldots x_N$ and $Y$ a finite number of values $y_1, \ldots y_M$. Letting $A_i = \{X = x_i\}$,

[1] This information is missing in the text of the exercise in Ref. [1]

$B_j = \{Y = y_j\}$, $i = 1, \dots N$, $j = 1, \dots M$, and denoting $\mathbb{I}_A$ the indicator function of the set $A$, we have

$$X = \sum_{i=1}^{N} x_i \mathbb{I}_{A_i}, \quad Y = \sum_{j=1}^{M} y_j \mathbb{I}_{B_j}.$$

Hence

$$XY = \sum_{i=1}^{N} \sum_{j=1}^{M} x_i y_j \mathbb{I}_{A_i} \mathbb{I}_{B_j} = \sum_{i=1}^{N} \sum_{j=1}^{M} x_i y_j \mathbb{I}_{A_i \cap B_j}$$

Hence, by the linearity of the expectation, and the assumed independence of $X, Y$,

$$\begin{aligned} \mathbb{E}[XY] &= \sum_{i=1}^{N} \sum_{j=1}^{M} x_i y_j \mathbb{E}[\mathbb{I}_{A_i \cap B_j}] \\ &= \sum_{i=1}^{N} \sum_{j=1}^{M} x_i y_j \mathbb{P}(A_i \cap B_j) \\ &= \sum_{i=1}^{N} \sum_{j=1}^{M} x_i y_j \mathbb{P}(A_i) \mathbb{P}(B_j) \\ &= \sum_{i=1}^{N} x_i \mathbb{P}(A_i) \sum_{j=1}^{M} \mathbb{P}(B_j) = \mathbb{E}[X]\mathbb{E}[Y]. \end{aligned}$$

As an example of uncorrelated, but not independent, random variables $X, Y$, consider

$$X = \begin{cases} -1 & \text{with prob. } 1/3 \\ 0 & \text{with prob. } 1/3 \\ 1 & \text{with prob. } 1/3 \end{cases} \qquad Y = X^2.$$

The random variables $X, Y$ are not independent, since $Y$ is not a deterministic constant (see Exercise 5.6 above). Moreover $XY = X^3 = X$ and thus $\mathbb{E}[XY] = \mathbb{E}[X^3] = \mathbb{E}[X] = 0$. Since $\mathbb{E}[X]\mathbb{E}[Y] = 0$, then $\mathrm{Cov}(X, Y) = 0$, i.e., the two random variables are uncorrelated.

To prove the inequality we first we notice that

$$\mathrm{Var}[\alpha X] = \mathbb{E}[\alpha^2 X^2] - \mathbb{E}[\alpha X]^2 = \alpha^2 \mathbb{E}[X^2] - \alpha^2 \mathbb{E}[X]^2 = \alpha^2 \mathrm{Var}[X],$$

$$\mathrm{Cov}(\alpha X, Y) = \mathbb{E}[\alpha XY] - \mathbb{E}[\alpha X]\mathbb{E}[Y] = \alpha \mathrm{Cov}(X, Y)$$

and

$$\begin{aligned} \mathrm{Var}[X + Y] = \mathbb{E}[(X + Y)^2] - \mathbb{E}[X + Y]^2 &= \mathbb{E}[X^2] + \mathbb{E}[Y^2] + 2\mathbb{E}[XY] \\ &- \mathbb{E}[X]^2 - \mathbb{E}[Y]^2 - 2\mathbb{E}[X]\mathbb{E}[Y] = \mathrm{Var}[X] + \mathrm{Var}[Y] + 2\mathrm{Cov}(X, Y). \end{aligned}$$

Hence letting $a \in \mathbb{R}$ we have

$$\mathrm{Var}[Y - aX] = a^2 \mathrm{Var}[X] + \mathrm{Var}[Y] - 2a\mathrm{Cov}(X, Y).$$

Since the variance of a random variable is always non-negative, the parabola

$$y(a) = a^2 \mathrm{Var}[X] + \mathrm{Var}[Y] - 2a\mathrm{Cov}(X, Y)$$

must always lie above the $a$-axis, or touch it at one single point $a = a_0$. Hence

$$\mathrm{Cov}(X, Y)^2 - \mathrm{Var}[X]\mathrm{Var}[Y] \leq 0,$$

which proves (1). Moreover $\mathrm{Cov}(X, Y)^2 = \mathrm{Var}[X]\mathrm{Var}[Y]$ if and only if there exists $a_0$ such that $\mathrm{Var}[-a_0 X + Y] = 0$, i.e., $Y = a_0 X + b_0$, for some constant $b_0$. Note that $a_0 \neq 0$, otherwise $Y$ is a deterministic constant. Substituting in the definition of covariance, we see that $\mathrm{Cov}(X, a_0 X + b_0) = a_0 \mathrm{Var}[X]$. Hence if the right inequality in (1) is an equality we have

$$a_0 \mathrm{Var}[X] = \sqrt{\mathrm{Var}[X]\mathrm{Var}[a_0 X + b_0])}, \quad \text{i.e., } a_0 \mathrm{Var}[X] = |a_0| \mathrm{Var}[X],$$

and thus $a_0 > 0$. Similarly one shows that if the left inequality becomes an equality then $a_0 < 0$.

# Exercise 5.15, Ref. [1]

Let $T > 0$ and $n \in \mathbb{N}$ be given. Define the stochastic process

$$\{W_n(t)\}_{t \in [0,T]}, \quad W_n(t) = \frac{1}{\sqrt{n}} M_{[nt]}, \tag{2}$$

where $[z]$ denotes the greatest integer smaller than or equal to $z$ and $M_k = X_1 + X_2 + \cdots + X_k$, $k = 1, \ldots, N$, is a symmetric random walk. It is assumed that the stochastic process $(X_1, \ldots, X_N)$ is defined for $N > [nT]$, so that $W_n(t)$ is defined for all $t \in [0, T]$. Compute $\mathbb{E}[W_n(t)]$, $\mathrm{Var}[W_n(t)]$, $\mathrm{Cov}[W_n(t), W_n(s)]$. Show that $\mathrm{Var}(W_n(t)) \to t$ and $\mathrm{Cov}(W_n(t), W_n(s)) \to \min(s, t)$ as $n \to +\infty$.

## Solution

By linearity of the expectation,

$$\mathbb{E}[W_n(t)] = \frac{1}{\sqrt{n}} \mathbb{E}[M_{[nt]}] = 0,$$

where we used the fact that $\mathbb{E}[X_k] = \mathbb{E}[M_k] = 0$. Since $\mathrm{Var}[M_k] = k$, we obtain

$$\mathrm{Var}[W_n(t)] = \frac{[nt]}{n}.$$

Since $nt \sim [nt]$, as $n \to \infty$, then $\lim_{n \to \infty} \mathrm{Var}[W_n(t)] = t$. As to the covariance of $W_n(t)$ and $W_n(s)$ for $s \neq t$, we compute

$$\begin{aligned}
\mathrm{Cov}[W_n(t), W_n(s)] &= \mathbb{E}[W_n(t)W_n(s)] - \mathbb{E}[W_n(t)]\mathbb{E}[W_n(s)] = \mathbb{E}[W_n(t)W_n(s)] \\
&= \mathbb{E}\left[\frac{1}{\sqrt{n}} M_{[nt]} \frac{1}{\sqrt{n}} M_{[ns]}\right] = \frac{1}{n} \mathbb{E}[M_{[nt]} M_{[ns]}].
\end{aligned} \tag{3}$$

Assume $t > s$ (a similar argument applies to the case $t < s$). If $[nt] = [ns]$ we have $\mathbb{E}[M_{[nt]}M_{[ns]}] = \mathrm{Var}[M_{[ns]}] = [ns]$. If $[nt] \geq 1 + [ns]$ we have

$$\mathbb{E}[M_{[nt]}M_{[ns]}] = \mathbb{E}[(M_{[nt]}-M_{[ns]})M_{[ns]}]+\mathbb{E}[M_{[ns]}^2] = \mathbb{E}[M_{[nt]}-M_{[ns]}]\mathbb{E}[M_{[ns]}]+\mathrm{Var}[M_{[ns]}] = [ns],$$

where we used that the increment $M_{[nt]} - M_{[ns]}$ is independent of $M_{[ns]}$. Replacing into (3) we obtain

$$\mathrm{Cov}[W_n(t), W_n(s)] = \frac{[ns]}{n}.$$

It follows that $\lim_{n\to\infty} \mathrm{Cov}[W_n(t), W_n(s)] = s$.

# Exercise 5.25, Ref. [1]

Let $\{W(t)\}_{t\in[0,T]}$ be a Brownian motion. Show that $\mathrm{Cov}[W(s), W(t)] = \min(s, t)$, for all $s, t \in [0, T]$. (Compare this with Exercise 5.15)

## Solution

As $\mathbb{E}[W(t)] = 0$ for all $t \geq 0$,

$$\mathrm{Cov}[W(s), W(t)] = \mathbb{E}[W(s)W(t)].$$

Assume $t > s$ (for $t < s$ the argument is identical). Using that the increments $W(t) - W(s)$ and $W(s) = W(s) - W(0)$ are independent we have

$$\begin{aligned}\mathbb{E}[W(s)W(t)] &= \mathbb{E}[W(s)(W(t) - W(s))] + \mathbb{E}[W(s)^2] \\ &= \mathbb{E}[W(s)]\mathbb{E}[W(t) - W(s)] + \mathrm{Var}[W(s)] = \mathrm{Var}[W(s)] = s.\end{aligned}$$