Exam for the course "Options and Mathematics" (CTH[*MVE095*], GU[*MMA700*])

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REMARK: No aids permitted

1. **Theorems.**

   Assume that the price $S(t)$ of a stock follows a N-period binomial model with parameters u, d, p and that the interest rate of the risk-free asset is a constant r. Show that this market is arbitrage-free if and only if $d < r < u$ (max. 3 points). Compute the expectation of $S(N)$ in the probability p and in the risk-neutral probability (max. 2 points).

2. (**Black-Scholes.**) A standard European derivative pays the amount $Y = (S(T) - S(0))_+$ at time of maturity T. Find the Black-Scholes price $\Pi_Y(0)$ of this derivative at time $t = 0$ assuming that the underlying stock pays the dividend $(1 - e^{-rT})S(\frac{T}{2}-)$ at time $t = \frac{T}{2}$ (max. 3 points). Compute the probability of positive return for a constant portfolio which is short 1 share of the derivative and short $S(0)e^{-rT}$ shares of the risk-free asset(max. 2 points).

   Solution. Letting $a = 1 - e^{-rT}$, the price at time $t = 0$ of a European derivative with pay-off function $g(x) = (x - S(0))_+$ is

$$
\begin{aligned}
\Pi_Y(0) &= e^{-rT}\int_{\mathbb{R}^3} g((1-a)S(0)e^{(r-\sigma^2/2)T+\sigma\sqrt{T}y})e^{-\frac{y^2}{2}}\frac{dy}{\sqrt{2\pi}} \\
&= e^{-rT}S(0)\int_{\mathbb{R}^3}(e^{-\frac{\sigma^2}{2}T+\sigma\sqrt{T}y} - 1)_+ e^{-\frac{y^2}{2}}\frac{dy}{\sqrt{2\pi}} \\
&= e^{-rT}S(0)\int_{\sigma\sqrt{T}/2}^{+\infty}\left(e^{-\frac{1}{2}(y-\sigma\sqrt{T})^2} - e^{\frac{y^2}{2}}\right)\frac{dy}{\sqrt{2\pi}} \\
&= e^{-rT}S(0)[\Phi(\frac{\sigma\sqrt{T}}{2}) - \Phi(-\frac{\sigma\sqrt{T}}{2})] \\
&= S(0)e^{-rT}(2\Phi(\frac{\sigma\sqrt{T}}{2}) - 1)
\end{aligned}
$$

   where Φ is the standard normal cumulative distribution (3 points). The value of the given portfolio at times $t = 0, T$ is

$$V(T) = -g(S(T)) - S(0)e^{-rT}e^{rT} = -(S(T) - S(0))_+ - S(0)$$

$$V(0) = -\Pi_Y(0) - S(0)e^{-rT} = -2S(0)e^{-rT}\Phi(\frac{\sigma\sqrt{T}}{2})$$

Hence $R(T) = V(T) - V(0)$ is given by

$$R(T) = \begin{cases} -S(T) + 2S(0)e^{-rT}\Phi(\sigma\sqrt{T}/2) & \text{if } S(T) > S(0) \\ S(0)[2e^{-rT}\Phi(\sigma\sqrt{T}/2) - 1] & \text{if } S(T) < S(0) \end{cases}$$

In particular, for $S(T) > S(0)$ we have

$$R(T) < S(0)[2e^{-rT}\Phi(\sigma\sqrt{T}/2) - 1], \quad \text{for } S(T) > S(0).$$

It follows that for $2e^{-rT}\Phi(\sigma\sqrt{T}/2) - 1 \leq 0$, the portfolio return is always negative. For $2e^{-rT}\Phi(\sigma\sqrt{T}/2) - 1 > 0$ we have

$$\begin{aligned}\mathbb{P}(R(T) > 0) &= \mathbb{P}(S(T) > S(0)) + \mathbb{P}(S(0) < S(T) < 2S(0)e^{-rT}\Phi(\sigma\sqrt{T}/2)) \\ &= \mathbb{P}(S(T) < 2S(0)e^{-rT}\Phi(\sigma\sqrt{T}/2))\end{aligned}$$

As

$$S(T) = (1-a)S(0)e^{(r-\sigma^2/2)T+\sigma\sqrt{T}G} = S(0)^{-\frac{\sigma^2}{2}T+\sigma\sqrt{T}G},$$

where G is a standard normal random variable, then

$$S(T) < 2S(0)e^{-rT}\Phi(\sigma\sqrt{T}/2) \Leftrightarrow G \leq \delta$$

where

$$\delta = \frac{(\frac{\sigma^2}{2} - r)T + \log(2\Phi(\sigma\sqrt{T}/2))}{\sigma\sqrt{T}}.$$

Hence

$$\mathbb{P}(S(T) < 2S(0)e^{-rT}\Phi(\sigma\sqrt{T}/2)) = \mathbb{P}(G \leq \delta) = \Phi(\delta).$$

(2 points)

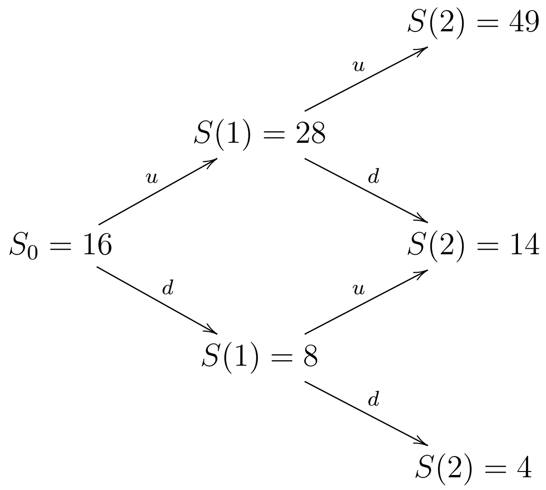
3. (**Binomial model.**). A compound option is an option whose underlying is another option. For instance, given $T_2 > T_1 > 0$ and $K_1, K_2 > 0$, a **call on a put** with maturity T_1 and strike K_1 is a contract that gives to its owner the right to buy at time T_1 for the price K_1 a put option on a stock with maturity T_2 and strike K_2.

   Let $S(t)$ be the price of the underlying stock of the put option. Assume that $S(t)$ follows a 2-period binomial model with parameters

   $$e^u = \frac{7}{4}, \quad e^d = \frac{1}{2}, \quad e^r = \frac{9}{8}, \quad p = \frac{1}{4}, \quad S(0) = 16$$

   Assume further that $T_2 = 2$, $T_1 = 1$, $K_1 = \frac{23}{9}$, $K_2 = 12$. Compute the initial price of the call on the put (max. 3 points). Compute the expected return for the owner of 1 share of the call on the put (max. 2 points).

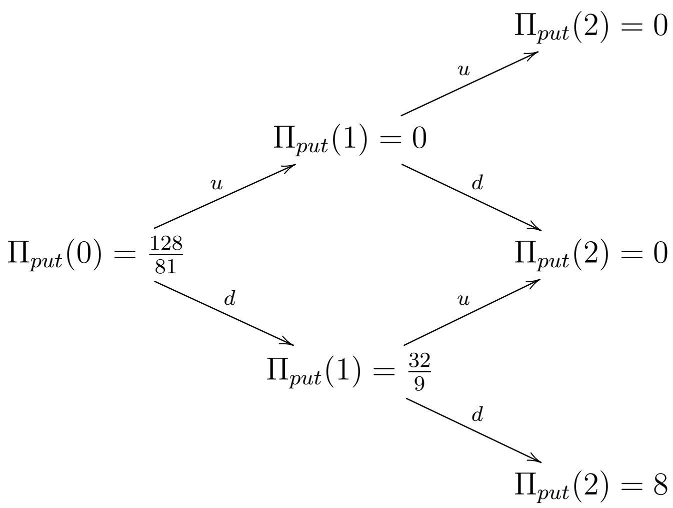
Solution. The binomial tree for the price of the stock is



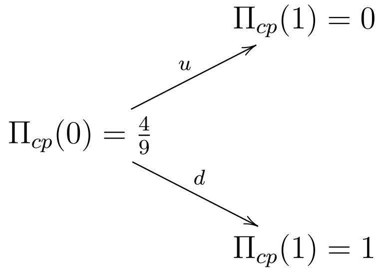
Moreover $q_u = \frac{e^r - e^d}{e^u - e^d} = 1/2 = q_d$. Using the recurrence formula $\Pi_Y(2) = Y$,

$$\Pi_Y(t) = e^{-r}(q_u \Pi_Y^u(t+1) + q_d \Pi_Y^d(t+1)), \quad t = 0, 1,$$

we find that the price $\Pi_{put}(t)$ of the put option is



The pay-off for the call on the put at time $t = T_1 = 1$ is $(\Pi_{put} - K_1)_+$. Note that since $\Pi_{put}(1) = g(S(1))$, then the call on the put can be treated as a standard European derivative on the stock expiring at time $T_1 = 1$. Hence the price $\Pi_{cp}(t)$ of the call on the put is



This completes the first part of the exercise (3 points). Now, the return of a portfolio with +1 share of the call on the put is path dependent. We have

$$R(u,u) = 0 - \frac{4}{9} = -\frac{4}{9}, \quad R(u,d) = 0 - \frac{4}{9} = -\frac{4}{9}$$

$$R(d,u) = 0 - \frac{4}{9} - \frac{23}{9} = -3, \quad R(d,d) = 8 - \frac{4}{9} - \frac{23}{9} = 5.$$

Hence the expected return is

$$\mathbb{E}[R] = p_u^2 R(u,u) + p_u p_d R(u,d) + p_d p_u R(d,u) + p_d^2 R(d,d) = \frac{77}{36} \approx 214\%$$