Hints for recommended exercises in Ref. [1], Ch. 5

Exercise 5.1

Solution:

$$
\begin{aligned}
2^{\Omega_2} = \Big\{ &\emptyset \text{ (set containing zero elements)}, \\
&\{(H,H)\}, \{(H,T)\}, \{(T,H)\}, \{(T,T)\} \text{ (set containing 1 element)}, \\
&\{(H,H),(H,T)\}, \{(H,H),(T,H)\}, \{(H,H),(T,T)\}, \{(H,T),(T,H)\}, \{(H,T),(T,T)\}, \\
&\qquad \{(T,H),(T,T)\}\} \text{ (set containing 2 elements)}, \\
&\{(H,H),(H,T),(T,H)\}, \{(H,H),(H,T),(T,T)\}, \{(H,H),(T,H),(T,T)\}, \\
&\qquad \{(H,T),(T,H),(T,T)\} \text{ (set containing 3 elements)}, \\
&\{(H,H),(H,T),(T,H),(T,T)\} = \Omega_2 \text{ (set containing 4 elements)} \Big\}
\end{aligned}
$$

Note that Ω_2 contains $2^2 = 4$ elements, hence 2^{Ω_2} contains $2^4 = 16$ elements.

Exercise 5.2

$$A = \{(H,H,T,T), (H,T,H,T), (H,T,T,H), (T,T,H,H), (T,H,T,H), (T,H,H,T)\}$$

and similarly for the other events.

Exercise 5.3

Using the result of Ex. 5.2:

$$
\begin{aligned}
\mathbb{P}(A) =& \mathbb{P}(\{(H,H,T,T)\}) + \mathbb{P}(\{(H,T,H,T)\}) + \mathbb{P}(\{(H,T,T,H)\}) + \mathbb{P}(\{(T,T,H,H)\}) \\
&+ \mathbb{P}(\{(T,H,T,H)\}) + \mathbb{P}(\{(T,H,H,T)\}) = 6p_H^2 p_T^2
\end{aligned}
$$

Moreover,

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$$

$A \cap B$ is the event that the number of heads equals the number of tails *and* successive tosses are different. From the form of A, see Ex. 5.2, we see that

$$A \cap B = \{(H,T,H,T), (T,H,T,H)\} = B.$$

Hence $\mathbb{P}(A|B) = 1$.

Exercise 5.4

By (5.10) we have

$$\mathbb{E}[X] = \sum_{\omega \in \Omega_N} (N_H(\omega) - N_T(\omega)) p^{N_H(\omega)} (1-p)^{N_T(\omega)}$$

Let $k = N_H(\omega)$; as $N_T(\omega) = N - k$, we have

$$\begin{aligned}\mathbb{E}[X] &= \sum_{k=0}^{N} \binom{N}{k} (2k - N) p^k (1-p)^{N-k} \\ &= (1-p)^N \Big(2 \sum_{k=0}^{N} \binom{N}{k} k \left(\frac{p}{1-p} \right)^k - N \sum_{k=0}^{N} \binom{N}{k} \left(\frac{p}{1-p} \right)^k \Big)\end{aligned}$$

By the binomial theorem the second sum is $1/(1-p)^N$. In the first sum we use the identity

$$k \binom{N}{k} = N \binom{N-1}{k-1},$$

which follows immediately by the definition of binomial coefficient. After simple calculations we obtain the final result

$$\mathbb{E}[X] = N(2p - 1).$$

Exercise 5.5

Follows easily by (5.11).