Hints for recommended exercises in Ref. [1] (week 5)

Exercise 5.9

We have

$$\mathrm{Cor}(X,Y) = \frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}[x]\mathrm{Var}[Y]}} = \frac{\mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]}{\sqrt{(\mathbb{E}[X^2] - \mathbb{E}[X]^2)(\mathbb{E}[Y^2] - \mathbb{E}[Y]^2)}}$$

To compute the expectations in the last member of the previous equation we note that, since $N_T(\omega) = 3 - N_H(\omega)$,

$$X(\omega) = 3 - 2N_H(\omega),\ Y = N_H(\omega),\ XY(\omega) = N_H(\omega)(3 - 2N_H(\omega)),$$
$$X(\omega)^2 = (3 - 2N_H(\omega))^2,\ Y(\omega)^2 = N_H(\omega)^2$$

In particular, all random variables for which we have to compute the expectation depend only on the number of heads $k = N_H(\omega) \in \{0,1,2,3\}$. Letting $Z(k)$ be any random variable on Ω_3 which depends only on k we have

$$\mathbb{E}[Z] = \sum_{k=0}^{3} \binom{3}{k} Z(k) p^k (1-p)^{3-k}$$

Exercise 5.10

Recall that $f_X(x) = \mathbb{P}(X = x)$. Writing

$$\{X = x\} = \cup_{y \in \mathrm{Im}(Y)} \{X = x, Y = y\},$$

then we obtain

$$\mathbb{P}(X = x) = \sum_{y \in \mathrm{Im}(Y)} \mathbb{P}(X = x, Y = y) = \sum_{y \in \mathrm{Im}(Y)} f_{X,Y}(x,y).$$

Here we used that $A \cap B = \emptyset \Rightarrow \mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B)$, which is immediate from the definition of probability, see def. 5.1. Remark: in the lecture notes there is a typo in ex. 5.10. Instead of $y \in \mathrm{Im}(Y)$ I wrote $y \in \mathrm{Im}(X)$ in the above formula.

Exercise 5.11

Create a table for X^2 and Y^2 and compute $\mathbb{E}[X^2], \mathbb{E}[Y^2]$ as in the example in the lecture notes. Then you get the variance by the formulas

$$\mathrm{Var}[X] = \mathbb{E}[X^2] - \mathbb{E}[X]^2, \quad \mathrm{Var}[Y] = \mathbb{E}[Y^2] - \mathbb{E}[Y]^2$$

and then compute $\mathrm{Cor}(X,Y) = \mathrm{Cov}(X,Y)/\sqrt{\mathrm{Var}[X]\mathrm{Var}[Y]}$.

Exercise 5.12

Just repeat the calculation above Exercise 5.12 in the lecture notes for every $y \in \mathrm{Im}(Y) = \{0, 6, 24, 60\}$.

Exercise 5.17

Take the expectation of (5.23) and use 4 of Theorem 5.2

Exercise 5.18

We have $\mathrm{Var}[S(N)] = \mathbb{E}_p[S(N)^2] - \mathbb{E}_p[S(N)]^2$. The expectation $\mathbb{E}_p[S(N)]$ is given in Theorem 5.4, hence we only have to compute $\mathbb{E}_p[S(N)^2]$. But

$$S(N)^2 = S(0)^2 \exp(2X_1 + \ldots 2X_N) = S(0)^2 \exp(Y_1 + \ldots Y_N),$$

where

$$Y_i = 2X_i = \begin{cases} 2u & \text{with prob. } p \\ 2d & \text{with prob. } (1-p) \end{cases}$$

Using the result of Theorem 5.4 with u, d replaced by $2u, 2d$ and $S(0)$ with $S(0)^2$ we find

$$\mathbb{E}_p[S(N)^2] = S(0)^2 (e^{2u} p + e^{2d}(1-p))^N$$

Exercise 5.19

The price of the stock up to time t, i.e., $S(0), \ldots S(t)$, is completely determined by the first t-tosses in ω.