Hints for recommended exercises in Ref. [1] (week 6)

Exercise 5.22

We have for example

$$\mathbb{E}[X] = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{\mathbb{R}} x e^{-\frac{(x-m)^2}{2\sigma^2}}\, dx$$

To compute the integral we write $x = (x - m) + m$. Hence

$$\frac{1}{\sqrt{2\pi\sigma^2}} \int_{\mathbb{R}} x e^{-\frac{(x-m)^2}{2\sigma^2}}\, dx = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{\mathbb{R}} (x-m) e^{-\frac{(x-m)^2}{2\sigma^2}}\, dx + m \frac{1}{\sqrt{2\pi\sigma^2}} \int_{\mathbb{R}} e^{-\frac{(x-m)^2}{2\sigma^2}}\, dx$$

We make the change of variables $y = (x - m)/\sigma$ in both integrals and obtain

$$\mathbb{E}[X] = \frac{\sigma}{\sqrt{2\pi}} \int_{\mathbb{R}} y e^{-\frac{y^2}{2}}\, dy + m \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-\frac{y^2}{2}}\, dy$$

Now, the first integral is clearly zero. In fact upon the change of variable $y \to -y$ we obtain

$$\int_{\mathbb{R}} y e^{-\frac{y^2}{2}}\, dy = -\int_{\mathbb{R}} y e^{-\frac{y^2}{2}}\, dy$$

and so the integral is zero ($A = -A$ implies $A = 0$). The second integral is 1, i.e.,

$$\int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}\, dy = 1.$$

In fact the integrand function is the density of a standard normal variable and therefore it integrates to 1 over the real line.

Exercise 5.26

We have

$$\mathbb{P}(a < S(t) < b) = \int_a^b f_{S(t)}(x)\, dx,$$

where the density of $S(t)$ is given by (5.37) in the lecture notes. After the appropriate change of variables we find

$$\mathbb{P}(a < S(t) < b) = \Phi(B) - \Phi(A),$$

where $\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{y^2}{2}}\, dy$ is the standard normal distribution and

$$A = \frac{\log \frac{a}{S_0} - \alpha t}{\sigma\sqrt{t}}, \quad B = \frac{\log \frac{b}{S_0} - \alpha t}{\sigma\sqrt{t}}$$

Exercise 6.1

Taylor expand the functions η_h, ξ_h with respect to h.