Exam for the course "Options and Mathematics" (CTH[*MVE095*], GU[*MMA700*]). June 4^{th}, 2015

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REMARK: No aids permitted

1. **Theorems**

   a) Assume that the dominance principle holds. Prove the put-call parity (max. 2 points)

   b) Let $\{S(t)\}_{t\in[0,T]}$ be a geometric Brownian motion. Derive the probability density of $S(t)$ (max. 1 point)

   c) Derive the formula for the Black-Scholes price of a call option (max. 2 points)

   Solution: See Lecture Notes.

2. Assume that the price $S(t)$ of a stock follows a 2-period binomial model with parameters

$$e^u = \frac{7}{4}, \quad e^d = \frac{1}{2}, \quad S(0) = 1, \quad p = 3/4.$$

   Assume also that the interest rate of the risk-free asset is such that $e^r = 9/8$.

   a) Compute the fair price at $t = 0, 1, 2$ of an American put with strike $K = 3/4$ and maturity $T = 2$ (max. 1 point)

   b) Compute the fair price at $t = 0, 1, 2$ of a European call with strike $K = 3/4$ and maturity $T = 2$ (max. 1 point)

   c) A derivative $\mathcal{U}$ gives to its owner the right to convert $\mathcal{U}$ at time $t = 1$ into either the European call or the American put defined above. Compute the fair price of $\mathcal{U}$ at time $t = 0$ (max. 1 point)

   d) Describe the *optimal* strategy that the holder of $\mathcal{U}$ should follow (max. 1 point)

   e) Compute the expected value at time $t = 2$ of a portfolio containing one share of $\mathcal{U}$ at time $t = 0$ and assuming that the American put is not exercised at time $t = 1$ (max. 1 point).

Solution: Let $\hat{\Pi}_{put}(t)$ denote the price of the American put and $\Pi_{call}(t)$ denote the price of the European call for $t = 0, 1, 2$. At maturity we have $\hat{\Pi}_{put}(2) = (3/4 - S(2))_+$, $\Pi_{call}(2) = (S(2) - 3/4)_+$, while for $t = 0, 1$ we use the recurrence formulas

$$\hat{\Pi}_{put}(t) = \max[(3/4 - S(t))_+, e^{-r}(q_u \hat{\Pi}^u_{put}(t+1) + q_d \hat{\Pi}^d_{put}(t+1))],$$

$$\Pi_{call}(t) = e^{-r}(q_u \Pi^u_{call}(t+1) + q_d \Pi^d_{call}(t+1))$$

with $q_u = q_d = 1/2$. We obtain the following diagrams for the price of the two options:

$$\begin{tikzcd}
 & & \hat{\Pi}_{put}(2) = 0 \\
 & \hat{\Pi}_{put}(1) = 0 \arrow[ur, "u"] \arrow[dr, "d"] & \\
\hat{\Pi}_{put}(0) = \frac{1}{9} \arrow[ur, "u"] \arrow[dr, "d"] & & \hat{\Pi}_{put}(2) = 0 \\
 & \hat{\Pi}_{put}(1) = \frac{1}{4} \arrow[ur, "u"] \arrow[dr, "d"] & \\
 & & \hat{\Pi}_{put}(2) = \frac{1}{2}
\end{tikzcd}$$

$$\begin{tikzcd}
 & & \Pi_{call}(2) = \frac{37}{16} \\
 & \Pi_{call}(1) = \frac{13}{12} \arrow[ur, "u"] \arrow[dr, "d"] & \\
\Pi_{call}(0) = \frac{41}{81} \arrow[ur, "u"] \arrow[dr, "d"] & & \Pi_{call}(2) = \frac{1}{8} \\
 & \Pi_{call}(1) = \frac{1}{18} \arrow[ur, "u"] \arrow[dr, "d"] & \\
 & & \Pi_{call}(2) = 0
\end{tikzcd}$$

This concludes part a)-b) of the exercise (1+1 points). As to the price of $\mathcal{U}$, note that the pay-off of $\mathcal{U}$ at time $t = 1$ is $\max(\hat{\Pi}_{put}(1), \Pi_{call}(1))$. Therefore, denoting by $\Pi_{\mathcal{U}}(t)$, $t = 0, 1$, the fair price of $\mathcal{U}$, we have

$$\Pi^u_{\mathcal{U}}(1) = \Pi^u_{call}(1) = \frac{13}{12}, \quad \Pi^d_{\mathcal{U}}(1) = \hat{\Pi}^d_{put}(1) = \frac{1}{4}$$

Hence

$$\Pi_{\mathcal{U}}(0) = e^{-r}(q_u \Pi_{\mathcal{U}}^u(1) + q_d \Pi_{\mathcal{U}}^d(1)) = \frac{16}{27}$$

This concludes the second part of the exercise (1 point). As to the optimal strategy, the holder of $\mathcal{U}$ will convert $\mathcal{U}$ into the American put if the stock price goes down at time 1 and into the European call if the price goes up. In the first case the investor exercises the American put, as its value equals the intrinsic value. This answers question c) (1 point). Finally, let $V(t)$, $t = 0, 1, 2$, be the value a portfolio with one share of $\mathcal{U}$. Using $p = 3/4$ and $1 - p = 1/4$ we find

$$V(2) = \begin{cases} \frac{37}{16} & \text{with probability } \left(\frac{3}{4}\right)^2, \\ \frac{1}{8} & \text{with probability } \frac{3}{4}\frac{1}{4}, \\ 0 & \text{with probability } \frac{3}{4}\frac{1}{4}, \\ \frac{1}{2} & \text{with probability } \left(\frac{1}{4}\right)^2 \end{cases}$$

Hence

$$\mathbb{E}[V(2)] = \frac{37}{16}\frac{9}{16} + \frac{1}{8}\frac{3}{16} + \frac{1}{2}\frac{1}{16} = \frac{347}{256}.$$

This concludes the fourth part of the exercise (1 point).

3. Consider a 1+1 dimensional, arbitrage-free stock market, in which the stock price $S(t)$ follows a one-period binomial model and the risk-free asset has a constant interest rate. Let $\Pi_Y(t)$, $t = 0, 1$, be the fair price of a European derivative with pay-off $Y = g(S(1)) \geq 0$. Let $h = (h_S, h_Y)$ be a portfolio invested in h_S shares of the stock and h_Y shares of the derivative. Show that h is not an arbitrage (max. 5 points).

   Solution: The argument is very similar to the one used for binomial markets. We have

$$\begin{aligned} \Pi_Y(0) = e^{-r}(q_u \Pi_Y^u(1) + q_d \Pi_Y^d(1)) &= e^{-r}(q_u g(S(0)e^u) + q_d g(S(0)e^d)) \\ &= e^{-r}(q_u f(u) + q_d f(d)), \end{aligned}$$

   where $f(u) = g(S(0)e^u) \geq 0$ and $f(d) = g(S(0)e^d) \geq 0$. The value of the portfolio h at time 0 is then given by

$$V(0) = h_S S(0) + h_Y e^{-r}(q_u f(u) + q_d f(d)),$$

   while at time $t = 1$ we have

$$V^u(1) = h_S S(0)e^u + h_Y f(u), \quad V^d(1) = h_S S(0)e^d + h_Y f(d).$$

   The portfolio is an arbitrage if $V(0) = 0$, $V^u(1) \geq 0$, $V^d(1) \geq 0$ and at least one of $V^u(1), V^d(1)$ is strictly positive. Assume that h is an arbitrage. From $V(0) = 0$ we get

$$h_S S(0) = -h_Y e^{-r}(q_u f(u) + q_d f(d)).$$

Inserting into the inequalities $V^u(1) \geq 0$ and $V^d(1) \geq 0$ we obtain

$$\begin{aligned} &- h_Y e^{-r}(q_u f(u) + q_d f(d))e^u + h_Y f(u) \geq 0 \\ &- h_Y e^{-r}(q_u f(u) + q_d f(d))e^u + h_Y f(d) \geq 0 \end{aligned}$$

Hence

$$\begin{aligned} &h_Y[(1 - q_u e^{u-r})f(u) - e^{u-r} q_d f(d)] \geq 0 \\ &h_Y[(1 - q_d e^{d-r})f(d) - e^{d-r} q_u f(u)] \geq 0 \end{aligned}$$

Since one of the two inequalities should be strict, then $h_Y \neq 0$. Assume $h_Y > 0$ (a similar argument applies if $h_Y < 0$). Then we obtain

$$(1 - q_u e^{u-r})f(u) - e^{u-r} q_d f(d) \geq 0 \tag{1}$$
$$(1 - q_d e^{d-r})f(d) - e^{d-r} q_u f(u) \geq 0 \tag{2}$$

The purpose is now to show that if one of the above inequalities is strict, then the other cannot be verified. Assume that (2) is strict. Then we have

$$f(d) > \frac{e^{d-r} q_u f(u)}{1 - q_d e^{d-r}},$$

where we used that $d < r$ and so $1 - q_d e^{d-r} > 0$. Replacing in (1) we obtain

$$\begin{aligned} (1 - q_u e^{u-r})f(u) - e^{u-r} q_d f(d) <& f(u) \left[1 - q_u e^{u-r} - e^{u-r} \frac{q_u q_d e^{d-r}}{1 - q_d e^{d-r}}\right] \\ =& f(u) \frac{1 - q_d e^{d-r} - q_u e^{u-r}}{1 - q_d e^{d-r}}. \end{aligned}$$

Recalling the identity $q_u e^u + q_d e^d = e^r$, we obtain

$$(1 - q_u e^{u-r})f(u) - e^{u-r} q_d f(d) < 0$$

i.e., (1) is not satisfied. At the same fashion one proves that (2) is not satisfied when (1) is strict.