Exam for the course "Options and Mathematics" (CTH[*MVE095*], GU[*MMA700*]) 2015/16

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REMARK: No aids permitted

1. Assume that the dominance principle holds and that there exists a risk-free asset with constant interest rate r. Prove the following:

   - the put-call parity (max. 2 points)
   - if $r \geq 0$, the price of call options is non-decreasing with the time of maturity (max. 1 point)
   - the price of call options is convex in the strike price (max. 1 point)

   Define and explain the concept of optimal exercise time of American put options (max. 1 point).

   Solution: See Theorem 1.1 and Def. 1.1 in the lecture notes.

2. Let the price $S(t)$ of a stock be given by a N-period binomial model with parameters $u > 0$, $d < 0$, $0 < r < u$, $p \in (0,1)$ and let $\widehat{\Pi}(t)$ be the binomial price of an American put on the stock with strike $K > 0$ and maturity $T = N$. Express $\widehat{\Pi}(N-1)$ as a function of $S(N-1)$ (max. 2 points). Show that it is optimal to exercise the American put at time $t = N-1$ if and only if the price of the stock at this time satisfies

$$S(N-1) \leq K \frac{1 - e^{-r} q_d}{1 - e^{-r} q_d e^d} \quad \text{(max. 3 points).}$$

   Solution: By definition of binomial price of American put options we have

$$\widehat{\Pi}(N) = (K - S(N))_+, \quad \widehat{\Pi}(N-1) = \max[(K - S(N-1))_+, e^{-r}(q_u \widehat{\Pi}^u(N) + q_d \widehat{\Pi}^d(N))]$$

   Using that

$$\widehat{\Pi}^u(N) = (K - S(N-1)e^u)_+, \quad \widehat{\Pi}^d(N) = (K - S(N-1)e^d)_+,$$

we obtain $\widehat{\Pi}(N-1) = f(S(N-1))$, where

$$f(x) = \max[(K-x)_+, e^{-r}(q_u e^u (Ke^{-u} - x)_+ + q_d e^d (Ke^{-d} - x)_+)].$$

This concludes the fist part of the exercise (2 points). For the second part of the exercise, we recall that it is optimal to exercise the derivative at time $t = N-1$ if and only if $\widehat{\Pi}(N-1) = (K - S(N-1))_+$, i.e., if and only if the binomial price of the American put equals its intrinsic value. To see when this happens, we compute $\widehat{\Pi}(N-1)$ when $S(N-1)$ lies in the intervals

$$S(N-1) \in [0, Ke^{-u}] := I_1, \;\; S(N-1) \in [Ke^{-u}, K] := I_2,$$
$$S(N-1) \in [K, Ke^{-d}] := I_3, \;\; S(N-1) \in [Ke^{-d}, +\infty) := I_4$$

Using the formula $\widehat{\Pi}(N-1) = f(S(N-1))$ proved above, we see that, for $S(N-1) \in I_1$,

$$\widehat{\Pi}(N-1) = \max[K - S(N-1), e^{-r}(q_u e^u (Ke^{-u} - S(N-1)) + q_d e^d (Ke^{-d} - S(N-1)))].$$

Using $q_u + q_d = 1$ and $q_u e^u + q_d e^d = e^r$ we obtain

$$\widehat{\Pi}(N-1) = \max[K - S(N-1), Ke^{-r} - S(N-1)] = K - S(N-1), \quad \text{for } S(N-1) \in I_1.$$

Similarly, for $S(N-1) \in I_2$ we have

$$\begin{aligned}\widehat{\Pi}(N-1) &= \max[K - S(N-1), e^{-r} q_d e^d (Ke^{-d} - S(N-1))] \\ &= \begin{cases} K - S(N-1) & \text{for } S(N-1) \leq S_* \\ e^{-r} q_d e^d (Ke^{-d} - S(N-1)) & \text{for } S(N-1) > S_* \end{cases}\end{aligned}$$

where

$$S_* = K \frac{1 - e^{-r} q_d}{1 - e^{-r} q_d e^d}.$$

Treating similarly the cases $S(N-1) \in I_3$ and $S(N-1) \in I_4$ we find

$$\widehat{\Pi}(N-1) = \begin{cases} K - S(N-1) & \text{for } 0 < S(N-1) \leq S_* \\ e^{-r} q_d e^d (Ke^{-d} - S(N-1)) & \text{for } S_* < S(N-1) \leq Ke^{-d} \\ 0 & \text{for } S > Ke^{-d} \end{cases}$$

We conclude that it is optimal to exercise the American put at time $t = N-1$ if and only if $S(N-1) \leq S_*$, which completes the solution of the second part of the exercise (3 points).

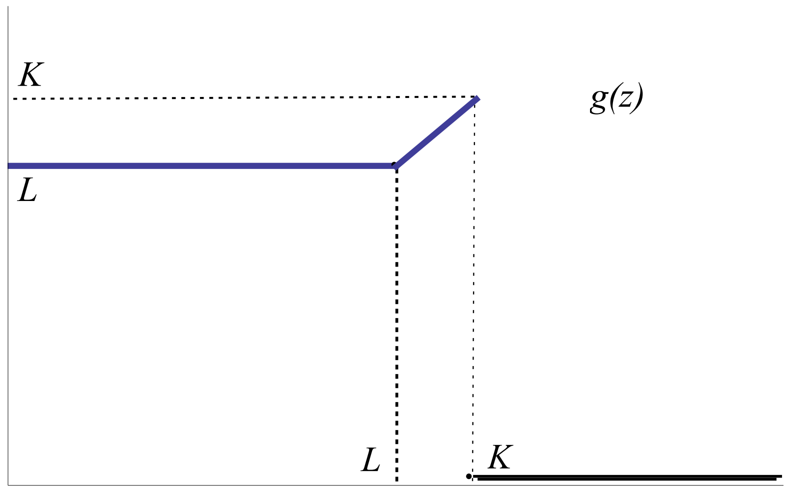
3. Let $0 < L < K$. A European style derivative on a stock with maturity $T > 0$ pays nothing to its owner when $S(T) > K$, while for $S(T) \leq K$ it lets the owner choose between 1 share of the stock and the fixed amount L. Draw the pay-off function of the derivative (max. 1 point). Compute the Black-Scholes price of the derivative (max.

2 points). Compute the number of shares of the stock in the hedging self-financing portfolio (max. 2 points).

Solution: The pay-off function is

$$g(z) = \begin{cases} L, & \text{for } 0 \leq z \leq L \\ z, & \text{for } L \leq z \leq K \\ 0, & \text{for } z > K, \end{cases}$$

which is depicted in the figure.



The Black-Scholes price of the derivative is given by $\Pi(t) = v(t, S(t))$, where

$$v(t,x) = \frac{e^{-r\tau}}{\sqrt{2\pi}} \int_{\mathbb{R}} g\left(xe^{(r-\frac{\sigma^2}{2})\tau} e^{\sigma\sqrt{\tau}y}\right) e^{-\frac{y^2}{2}} dy, \quad \tau = T - t,$$

where $\sigma > 0$ is the volatility of the stock and r is the interest rate of the risk-free asset. Replacing the pay-off function above we find:

$$\begin{aligned} v(t,x) &= \frac{e^{-r\tau}}{\sqrt{2\pi}} \int_{-\infty}^{d_1(L)} L e^{-\frac{y^2}{2}} dy + \frac{e^{-r\tau}}{\sqrt{2\pi}} \int_{d_1(L)}^{d_1(K)} xe^{(r-\frac{\sigma^2}{2})\tau} e^{\sigma\sqrt{\tau}y} e^{-\frac{y^2}{2}} dy \\ &= Le^{-r\tau}\Phi(d_1(L)) + x[\Phi(d_2(K)) - \Phi(d_2(L))], \end{aligned}$$

where $\Phi(z)$ is the standard normal distribution, $d_2(a) = d_1(a) - \sigma\sqrt{\tau}$ and

$$d_1(a) = \frac{\log\frac{L}{x} - (r - \frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}}.$$

This concludes the second part of the exercise (2 points). The number of shares of the stock in the hedging self-financing portfolio is $h_S(t) = \partial_x v(t, S(t))$. We use

$$\partial_x[\Phi(d_1(a))] = \phi(d_1(a))\partial_x[d_1(a)] = -\frac{\phi(d_1(a))}{x\sigma\sqrt{\tau}},$$

$$\partial_x[\Phi(d_2(a))] = -\frac{\phi(d_2(a))}{x\sigma\sqrt{\tau}}$$

where $\phi(z) = \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}}$ is the standard normal density function. Hence

$$\partial_x v(t,x) = -\frac{\phi(d_1(L))}{x\sigma\sqrt{\tau}} Le^{-r\tau} + \Phi(d_2(K)) - \Phi(d_2(L)) + \frac{\phi(d_2(L)) - \phi(d_2(K))}{\sigma\sqrt{\tau}}$$

The result can be further simplified by noticing that

$$\phi(d_2(L)) - \phi(d_1(L))\frac{L}{x}e^{-r\tau} = 0$$

(see also sec. 6.2 in the lecture notes). Hence we finally obtain

$$\partial_x v(t,x) = \Phi(d_2(K)) - \Phi(d_2(L)) - \frac{\phi(d_2(K))}{\sigma\sqrt{\tau}}.$$