

Exam for the course “Options and Mathematics”  
 (CTH[MVE095], GU[MMA700]). Period 4, 2013/14

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REMARK: No aids permitted

1. Consider a 3-period binomial asset pricing model with the following parameters:

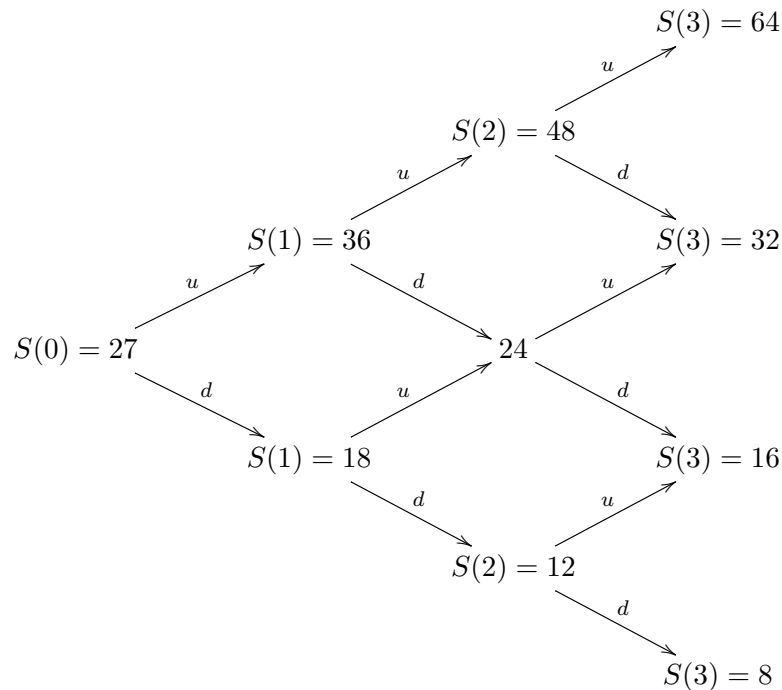
$$e^u = \frac{4}{3}, \quad e^d = \frac{2}{3}, \quad p = \frac{3}{4}.$$

Assuming  $S(0) = 27$  and that the risk-free asset has zero interest rate ( $r = 0$ ), compute the initial price of the Lookback Option with pay-off

$$Y = S(3) - \min(S(0), S(1), S(2), S(3))$$

and time of maturity  $T = 3$  (max. 3 points). Compute the probability that the derivative expires in the money (max. 1 point) and the probability that the return of a constant portfolio with a short position on this derivative be positive (max. 1 point).

**Solution:** The binomial tree of the stock price is



To compute the initial price of a non-standard European derivative it is convenient to use the formula

$$\Pi_Y(0) = e^{-rN} \sum_{x \in \{u,d\}^N} q_{x_1} \cdots q_{x_N} Y(x), \quad (1)$$

where  $Y(x)$  denotes the pay-off as a function of the path of the stock price. In this exercise we have  $N = 3$ ,  $r = 0$  and

$$q_u = q_d = \frac{1}{2}.$$

So, it remains to compute the pay-off for all possible paths of the stock price, where

$$Y = S(3) - \min(S(0), S(1), S(2), S(3)).$$

For instance

$$Y(u, u, u) = 64 - \min(27, 36, 48, 64) = 64 - 27 = 37.$$

Similarly we find

$$Y(u, u, d) = 5, \quad Y(u, d, u) = 8, \quad Y(u, d, d) = 0$$

$$Y(d, u, u) = 14, \quad Y(d, u, d) = 0, \quad Y(d, d, u) = 4, \quad Y(d, d, d) = 0.$$

Replacing in (1) we obtain

$$\begin{aligned} \Pi_Y(0) &= (q_u)^3 Y(u, u, u) + (q_u)^2 q_d Y(u, u, d) + (q_u)^2 q_d Y(u, d, u) \\ &\quad + (q_u)^2 q_d Y(d, u, u) + (q_d)^2 q_u Y(d, d, u) \\ &= \left(\frac{1}{2}\right)^3 (37 + 5 + 8 + 14 + 4) = \frac{17}{2} = 8.5. \end{aligned}$$

This concludes the first part of the exercise (3 points). The probability that the derivative expires in the money is the probability that  $Y > 0$ . Hence we just sum the probabilities of the paths which lead to a positive pay-off:

$$\begin{aligned} \mathbb{P}(Y > 0) &= \mathbb{P}(\{u, u, u\}) + \mathbb{P}(\{u, u, d\}) + \mathbb{P}(\{u, d, u\}) + \mathbb{P}(\{d, u, u\}) + \mathbb{P}(\{d, d, u\}) \\ &= p^3 + (1-p)p^2 + p^2(1-p) + p^2(1-p) + (1-p)^2 p \\ &= p^3 + 3p^2(1-p) + (1-p)^2 p = \frac{27}{64} + 3 \frac{9}{16} \frac{1}{4} + \frac{1}{16} \frac{3}{4} = \frac{57}{64} \approx 89\% \end{aligned}$$

This concludes the second part of the exercise (1 point). Next consider a constant portfolio with a short position on the derivative. This means that we buy the derivative at time  $t = 0$  and we wait (without changing the portfolio) until the expiration time  $t = 3$ . The return will be positive if  $\Pi_Y(3) < \Pi_Y(0)$ . But  $\Pi_Y(3) = Y$ , which, according to the computations above, is smaller than  $\Pi_Y(0) = 8.5$  when the stock price follows one of the paths  $\{u, u, d\}$ ,  $\{u, d, u\}$ ,  $\{u, d, d\}$ ,  $\{d, u, d\}$ ,  $\{d, d, u\}$  or  $\{d, d, d\}$ . Hence

$$\mathbb{P}[\Pi_Y(3) < \Pi_Y(0)] = 2p^2(1-p) + 3(1-p)^2 p + (1-p)^3 = \frac{7}{16} \approx 44\%$$

This concludes the third part of the exercise (1 point).

2. Compute the Black-Scholes price of European calls and puts (max. 3 points). Next consider a constant portfolio in the interval  $[0, T]$  which is invested in  $N$  standard European derivatives with pay-off functions  $g_1, \dots, g_N$  and time of maturity  $T$ . Let the price of these derivatives be given by the Black-Scholes formula. Show that the portfolio satisfies the dominance principle (max. 2 points).

**Solution:** See Borell's notes, Theorems 5.2.1 and 5.1.2

3. Consider two stocks with prices

$$S_1(t) = S_1(0)e^{\left(\mu_1 - \frac{\sigma_1^2}{2}\right)t + \sigma_1 W_1(t)}, \quad S_2(t) = S_2(0)e^{\left(\mu_2 - \frac{\sigma_2^2}{2}\right)t + \sigma_2 W_2(t)}$$

where  $\sigma_1, \sigma_2 > 0$ ,  $\mu_1, \mu_2 \in \mathbb{R}$  and  $W_1(t), W_2(t)$  are two Brownian motions. Let  $T > 0$  and assume that  $W_1(T)$  and  $W_2(T)$  are independent random variables. Compute the Markowitz portfolio of an investor with initial capital  $K > 0$  and risk aversion  $\theta$  who wants to invest in the stocks and in a money market with interest  $r > 0$  during the interval of time  $[0, T]$  (max. 5 points)

**Solution:** . Let  $a_1$  be the number of shares of the first stock and  $a_2$  the number of shares of the second stock in the Markowitz portfolio and let

$$\pi_1 = \frac{a_1 S_1(0)}{K}, \quad \pi_2 = \frac{a_2 S_2(0)}{K}.$$

Then we have the formula

$$\begin{pmatrix} \pi_1 \\ \pi_2 \end{pmatrix} = \frac{1}{2\theta} C^{-1} \begin{pmatrix} \mathbb{E}[R_1] - \rho \\ \mathbb{E}[R_2] - \rho \end{pmatrix} \quad (2)$$

where  $R_1, R_2$  are the returns of the two stocks,  $\rho = e^{rT} - 1$  and  $C^{-1}$  is the inverse of the covariant matrix  $C_{ij} = \text{Cov}(R_i, R_j)$ . The random variables  $R_1, R_2$ , are given by

$$R_1 = \frac{S_1(T) - S_1(0)}{S_1(0)}, \quad R_2 = \frac{S_2(T) - S_2(0)}{S_2(0)}.$$

It follows that

$$\mathbb{E}[R_1] = \frac{1}{S_1(0)} \mathbb{E}[S_1(T)] - 1, \quad \text{Var}[R_1] = \frac{1}{S_1(0)^2} \text{Var}[S_1(T)]$$

and similarly for the second stock. Since the expectation and the variance of the geometric Brownian motion  $S_1(t)$  are given by

$$\mathbb{E}[S_1(t)] = S_1(0)e^{\mu_1 t}, \quad \text{Var}[S_1(t)] = S_1(0)^2 e^{2\mu_1 t} (e^{\sigma_1^2 t} - 1)$$

we obtain

$$\mathbb{E}[R_1] = e^{\mu_1 T} - 1, \quad \text{Var}[R_1] = e^{2\mu_1 T} (e^{\sigma_1^2 T} - 1).$$

and similarly for the second stock

$$\mathbb{E}[R_2] = e^{\mu_2 T} - 1, \quad \text{Var}[R_2] = e^{2\mu_2 T} (e^{\sigma_2^2 T} - 1).$$

Since  $W_1(T)$  and  $W_2(T)$  are independent, then  $R_1$  and  $R_2$  are also independent. Therefore  $\text{Cov}(R_1, R_2) = C_{12} = C_{21} = 0$  and so the matrix of covariances of the random variables  $R_1, R_2$  is given by

$$C = \begin{pmatrix} \text{Var}[R_1] & 0 \\ 0 & \text{Var}[R_2] \end{pmatrix} \Rightarrow C^{-1} = \begin{pmatrix} 1/\text{Var}[R_1] & 0 \\ 0 & 1/\text{Var}[R_2] \end{pmatrix}$$

Replacing in (2) we obtain

$$\pi_1 = \frac{1}{2\theta} \frac{\mathbb{E}[R_1] - \rho}{\text{Var}[R_1]} = \frac{1}{2\theta} \frac{e^{\mu_1 T} - e^{rT}}{e^{2\mu_1 T}(e^{\sigma_1^2 T} - 1)}, \quad \pi_2 = \frac{1}{2\theta} \frac{\mathbb{E}[R_2] - \rho}{\text{Var}[R_2]} = \frac{1}{2\theta} \frac{e^{\mu_2 T} - e^{rT}}{e^{2\mu_2 T}(e^{\sigma_2^2 T} - 1)}$$

This completes the exercise (5 points).