Exercise 3, Chapter 3, Appendix D

A European derivative with expiration $T = N$ pays the amount $Y = \log(S(T)/S(0))$. Find $\Pi_Y(0)$

Solution:

$$\Pi_Y(0) = e^{-rN} \sum_{x\in\{u,d\}^N} (q_u)^{N_u(x)}(q_d)^{N_d(x)} \log(S(N)/S(0)),$$

by definition 3.3 (at $t = 0$). Replacing $S(N) = S(0)e^{N_u(x)u+N_d(x)d}$ we obtain

$$\Pi_Y(0) = e^{-rN} \sum_{x\in\{u,d\}^N} (q_u)^{N_u(x)}(q_d)^{N_d(x)}(N_u(x)u + N_d(x)d).$$

Replacing $N_d(x) = N - N_u(x)$ we obtain

$$\begin{aligned}\Pi_Y(0) &= e^{-rN}(q_d)^N \sum_{x\in\{u,d\}^N} (\frac{q_u}{q_d})^{N_u(x)}(N_u(x)(u-d) + Nd)\\ &= e^{-rN}(q_d)^N(u-d) \sum_{x\in\{u,d\}^N} (\frac{q_u}{q_d})^{N_u(x)}N_u(x) + Nde^{-rN}(q_d)^N \sum_{x\in\{u,d\}^N} (\frac{q_u}{q_d})^{N_u(x)}\end{aligned}$$

Letting $N_u(x) = k$ we obtain

$$\Pi_Y(0) = e^{-rN}(q_d)^N(u-d)\sum_{k=0}^{N}\binom{N}{k}k(\frac{q_u}{q_d})^k + Nde^{-rN}(q_d)^N\sum_{k=0}^{N}\binom{N}{k}(\frac{q_u}{q_d})^k \tag{1}$$

Using the binomial theorem, the second sum is

$$\sum_{k=0}^{N}\binom{N}{k}(\frac{q_u}{q_d})^k = (1+\frac{q_u}{q_d})^N = \frac{1}{(q_d)^N}$$

In the first sum we use the identity

$$k\binom{N}{k} = N\binom{N-1}{k-1}.$$

Hence

$$\begin{aligned}
\sum_{k=0}^{N}\binom{N}{k}k(\frac{q_u}{q_d})^k &= \sum_{k=1}^{N}\binom{N}{k}k(\frac{q_u}{q_d})^k \\
&= N\sum_{k=1}^{N}\binom{N-1}{k-1}(\frac{q_u}{q_d})^k = N\sum_{j=0}^{N-1}\binom{N-1}{j}(\frac{q_u}{q_d})^{j+1} \\
&= N\frac{q_u}{q_d}\sum_{j=0}^{N-1}\binom{N-1}{j}(\frac{q_u}{q_d})^j = N\frac{q_u}{q_d}(1+\frac{q_u}{q_d})^{N-1} \\
&= N\frac{q_u}{(q_d)^N}
\end{aligned}$$

Replacing in (1) we find

$$\begin{aligned}
\Pi_Y(0) &= e^{-rN}(q_d)^N(u-d)N\frac{q_u}{(q_d)^N} + Nde^{-rN}(q_d)^N\frac{1}{(q_d)^N} \\
&= Ne^{-rN}(q_u(u-d)+d) = Ne^{-rN}(q_uu+d(1-q_u)) = Ne^{-rN}(q_uu+q_dd).
\end{aligned}$$