Summary of the course "Options and Mathematics"

The following is a brief summary of the course "options and mathematics" with a list of the most important formulas to be remembered for the exam.

REMARKS

- this text cannot be brought to the exam!
- A list of definitions/theorems to know for the exam can be found in the course homepage. The present text contains most (but not all!) the formulas that appear in the definitions/theorems in this list
- The purpose of this text is only to help the students to check whether they have learned all the relevant concepts presented in the course. It is *not* a substitute for the lecture notes.

1 Introduction

The course deals with the basics of "options pricing theory", i.e., the theory which aims to define and study the concept of fair price of options (and other financial derivatives). Options are divided into European options (exercise is possible only at maturity) and American options (earlier exercise is allowed). Options are further divided into standard options (the pay-off depends only on the price of the underlying at the exercising time, e.g., call, put and digital options) and non-standard options (e.g., compound options, lookback options, Asian options, etc.).

2 Dominance principle

Some qualitative properties of European call and put options can be derived by first principles, such as the dominance principle. Among these properties is the put-call parity identity, which is

$$S(t) - C(t, S(t), K, T) = Ke^{-r(T-t)} - P(t, S(t), K, T) \tag{1}$$

where $C(t, S(t), K, T)$ is the price at time t of the European call with maturity T and strike K; $S(t)$ is the price at time t of the underlying stock and $P(t, S(t), K, T)$ is the price of the corresponding European put. The pay-off of the call and the put are given by

$$Y_{\text{call}} = C(T, S(T), K, T) = (S(T) - K)_+, \quad Y_{\text{put}} = P(T, S(T), K, T) = (K - S(T))_+ \tag{2}$$

where $(x)_+ = \max(x, 0)$. An optimal exercise time for the American put is a time t at which

$$\widehat{P}(t, S(t), K, T) = (K - S(t))_+. \tag{3}$$

3 Binomial model

A binomial market is a market that consists of a stock with price $S(t)$ and a risk-free asset with price $B(t)$ given by

$$S(t) = \begin{cases} S(t-1)e^u & \text{with prob. } p \\ S(t-1)e^d & \text{with prob. } 1-p \end{cases} \qquad B(t) = B(0)e^{rt} \tag{4}$$

where

$$t \in \mathcal{I} = \{1, \dots, N\}, \quad p \in (0,1), \quad d < r < u \text{ (arbitrage-free condition)} \tag{5}$$

The prices $S(0), B(0)$ at time $t = 0$ are given positive constants.

A portfolio process $\{(h_S(t), h_B(t))\}_{t\in\mathcal{I}}$ is self-financing if

$$h_S(t)S(t-1) + h_B(t)B(t-1) = V(t-1), \quad t \in \mathcal{I}, \tag{6}$$

where

$$V(t) = h_S(t)S(t) + h_B(t)B(t) \tag{7}$$

is the portfolio value at time t. $(h_S(t), h_B(t))$ is the portfolio position in the interval $(t-1, t]$. We always assume $h_S(0) = h_S(1)$, $h_B(0) = h_B(1)$.

The so-called risk-neutral probability is defined by

$$q_u = \frac{e^r - e^d}{e^u - e^d}, \quad q_d = 1 - q_u \quad (q_u, q_d \in (0,1)). \tag{8}$$

The value $V(t)$ of self-financing portfolio processes satisfies the formulas:

$$V(t) = e^{-r}[q_u V^u(t+1) + q_d V^d(t+1)], \tag{9}$$

$$V(t) = e^{-r(N-t)} \sum_{(x_{t+1},\dots,x_N)\in\{u,d\}^{N-t}} q_{x_{t+1}} \dots q_{x_N} V(N, x) \tag{10}$$

for $t = 0, \dots, N-1$; for $t = 0$ the second equation becomes

$$V(0) = e^{-rN} \sum_{x\in\{u,d\}^N} (q_u)^{N_u(x)}(q_d)^{N_d(x)} V(N, x) \tag{11}$$

Here $N_u(x)$ is the numbers of u in x and $N_d(x) = N - N_u(x)$ is the number of d in x. Moreover

$$V^u(t) = h_S(t)S(t-1)e^u + h_B(t)B(t-1)e^r \tag{12}$$

$$V^d(t) = h_S(t)S(t-1)e^d + h_B(t)B(t-1)e^r \tag{13}$$

is the portfolio value at time t assuming that the stock price goes up (resp. down) at time t.

A portfolio process is said to be an arbitrage if its value $V(t)$ satisfies

1) $V(0) = 0$
2) $V(N, x) \geq 0$, for all $x \in \{u, d\}^N$
3) There exists $y \in \{u, d\}^N$ such that $V(N, y) > 0$

Binomial markets are arbitrage free if and only if $d < r < u$.

The binomial price of the European derivative with pay-off $Y = Y(x)$ and time of maturity $T = N$ is defined by

$$\Pi_Y(t) = e^{-r(N-t)} \sum_{(x_{t+1},\dots,x_N)\in\{u,d\}^{N-t}} q_{x_{t+1}} \dots q_{x_N} Y(x), \tag{14}$$

for $t = 0, \dots, N-1$, while

$$\Pi_Y(N) = Y. \tag{15}$$

When solving an exercise that asks to compute the binomial price of a European derivative, it is convenient to use the recurrence formula

$$\Pi_Y(t) = e^{-r}[q_u \Pi_Y^u(t+1) + q_d \Pi_Y^d(t+1)] \tag{16}$$

if the derivative is standard (i.e. $Y = g(S(N))$), and the formula

$$\Pi_Y(0) = e^{-rN} \sum_{x \in \{u,d\}^N} (q_u)^{N_u(x)} (q_d)^{N_d(x)} Y(x) \tag{17}$$

when the derivative is non-standard (i.e., $Y = g(S(1), \ldots, S(N))$). (Exercises on non-standard derivatives ask to compute the price only at time $t = 0$.)

A portfolio process $\{(h_S(t), h_B(t))\}_{t \in \mathcal{I}}$ is said to be hedging the derivative if its value satisfies $V(N, x) = Y(x)$, for all $x \in \{u, d\}^N$. It is called predictable if $(h_S(t), h_B(t)) = H_t(S_0, \ldots, S(t-1))$ for some functions $H_1, \ldots, H_N$. The portfolio given by

$$h_S(t) = \frac{1}{S(t-1)} \frac{\Pi_Y^u(t) - \Pi_Y^d(t)}{e^u - e^d} \tag{18}$$

$$h_B(t) = \frac{e^{-r}}{B(t-1)} \frac{e^u \Pi_Y^d(t) - e^d \Pi_Y(t)}{e^u - e^d}, \tag{19}$$

for $t \in \mathcal{I}$ (and $h_S(1) = h_S(0), h_B(1) = h_B(0)$) is a self-financing, predictable, hedging portfolio for the standard European derivative with pay-off $Y = g(S(N))$ at maturity $T = N$.

The binomial price $\widehat{\Pi}_Y(t)$ of the American derivative with intrinsic value $Y(t) = g(S(t))$ and maturity $T = N$ is given by the recurrence formula

$$\widehat{\Pi}_Y(N) = Y(N), \quad \widehat{\Pi}_Y(t) = \max[Y(t), e^{-r}(q_u \widehat{\Pi}_Y^u(t+1) + q_d \widehat{\Pi}_Y^d(t))], \; t = 0, \ldots, N-1. \tag{20}$$

A portfolio process $\{(h_S(t), h_B(t))\}_{t \in \mathcal{I}}$ is said to generate the cash flow $C(t-1)$ if the following holds for $t \in \mathcal{I}$:

$$h_S(t) S(t-1) + h_B(t) B(t-1) = V(t-1) - C(t-1). \tag{21}$$

A portfolio process is said to be hedging the American derivative with intrinsic value $Y(t)$ and maturity $T = N$ if its value satisfies $V(N) = Y(N)$ and $V(t) \geq Y(t)$ for $t = 0, \ldots, N-1$. The portfolio process given by

$$\widehat{h}_S(t) = \frac{1}{S(t-1)} \frac{\widehat{\Pi}_Y^u(t) - \widehat{\Pi}_Y^d(t)}{e^u - e^d} \tag{22}$$

$$\widehat{h}_B(t) = \frac{e^{-r}}{B(t-1)} \frac{e^u \widehat{\Pi}_Y^d(t) - e^d \widehat{\Pi}_Y^u(t)}{e^u - e^d}, \tag{23}$$

is a predictable, hedging portfolio for the American derivative and generates the cash flow

$$C(0) = 0, \quad C(t-1) = \widehat{\Pi}_Y(t-1) - e^{-r}[q_u \widehat{\Pi}_Y^u(t) + q_d \widehat{\Pi}_Y^d(t)]. \tag{24}$$

Remark: the formulas for the hedging portfolio in the European and American case are the same.

4 Probability theory

The binomial stock price can be interpreted as a stochastic process $\{S(t)\}_{t \in \mathcal{I}}$ defined on the N-coin toss probability space $(\Omega_N, \mathbb{P}_p)$. The discounted price of the stock is the stochastic process $\{S^*(t)\}_{t \in \mathcal{I}}$ given by

$$S^*(t) = e^{-rt} S(t). \tag{25}$$

The discounted stock price is a martingale in the measure $\mathbb{P}_p$ if and only if $d < r < u$ and $p = q = \frac{e^r - e^d}{e^u - e^d}$. A discrete stochastic process $\{X_1, X_2, \ldots\}$ is a martingale if and only if

$$\mathbb{E}[X_{n+1} | X_1, \ldots, X_n] = X_n, \quad \text{for all } n \geq 1 \tag{26}$$

The expectation of the binomial stock price is given by

$$\mathbb{E}_p[S(N)] = S(0)(e^u p + e^d(1-p))^N. \tag{27}$$

In particular, for $p = q$ (risk neutral measure) we obtain

$$\mathbb{E}_q[S(N)] = S(0)e^{rN}. \tag{28}$$

5 Black-Scholes price

The stochastic process $\{S(t)\}_{t\geq 0}$ given by

$$S(t) = S(0)e^{\alpha t + \sigma W(t)} \tag{29}$$

is called geometric Brownian motion with mean of log return $\alpha \in \mathbb{R}$ and volatility $\sigma > 0$. Here $\{W(t)\}_{t\geq 0}$ is a Brownian motion. It can be shown that, in the time continuum limit, the binomial stock price converges (in distribution) to the geometric Brownian motion. The density of the geometric Brownian motion is given by

$$f_{S(t)}(x) = \frac{\mathbb{I}_{x>0}}{\sqrt{2\pi\sigma^2 t}}\frac{1}{x}\exp\left(-\frac{(\log x - \log S(0) - \alpha t)^2}{2\sigma^2 t}\right) \tag{30}$$

The Black-Scholes price of the European derivative with pay-off $Y = g(S(T))$ and maturity $T > 0$ is given by $\Pi_Y(t) = v(t, S(t))$, where

$$v(t,x) = \frac{e^{-r\tau}}{\sqrt{2\pi}}\int_{\mathbb{R}} g(xe^{(r-\frac{\sigma^2}{2})\tau + \sigma\sqrt{\tau}y})e^{-\frac{y^2}{2}}\,dy, \quad \tau = T - t. \tag{31}$$

The number of shares in the hedging (self-financing) portfolio process are given by

$$h_S(t) = \Delta(t, S(t)), \quad \Delta(t,x) = \partial_x v(t,x), \quad h_B(t) = \frac{1}{B(t)}(\Pi_Y(t) - h_S(t)S(t)) \tag{32}$$

Remark: the formula for $h_B(t)$ follows by the replicating condition: $h_S(t)S(t) + h_B(t)B(t) = \Pi_Y(t)$.

For call options we denote $v(t,x) = C(t,x)$ and in this case we have

$$C(t,x) = x\Phi(d_1) - Ke^{-r\tau}\Phi(d_2), \quad d_1 = d_2 + \sigma\sqrt{\tau}, \quad d_2 = \frac{\log\frac{x}{K} + (r - \frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}} \tag{33}$$

where

$$\Phi(z) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z} e^{-\frac{y^2}{2}}\,dy \tag{34}$$

is the standard normal distribution. Moreover

$$\Delta(t,x) = \partial_x C(t,x) = \Phi(d_1) \tag{35}$$

For put options we denote $v(t,x) = P(t,x)$ and in this case we have

$$P(t,x) = \Phi(-d_2)Ke^{-r\tau} - \Phi(-d_1)x, \qquad \Delta(t,x) = \partial_x P(t,x) = -\Phi(-d_1). \tag{36}$$

The Black-Schole pricing formula $\Pi_Y(t) = v(t, S(T))$ can also be written in the probabilistic form

$$\Pi_Y(t) = e^{-r\tau}\mathbb{E}[g(S(t)e^{(r-\frac{\sigma^2}{2})\tau + \sigma(W(T)-W(t))})]. \tag{37}$$

In the case that the underlying stock pays the dividend $aS(t_0^-)$ at time $t_0 \in (0,T)$, where $a \in (0,1)$, the Black-Scholes price of the European derivative becomes

$$\Pi_Y^{(a,t_0)}(t) = \begin{cases} v(t,(1-a)S(t)), & \text{for } t < t_0, \\ v(t,S(t)) & \text{for } t \geq t_0 \end{cases} \tag{38}$$