Exam for the course "Options and Mathematics" (CTH[*MVE095*], GU[*MMA700*]) 2016/17

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REMARKS: (1) No aids permitted (2) Minor errors in the calculations will be forgiven, but remember that fractions look nicer when you simplify them!

1. (i) Give and justify the definition of Black-Scholes price (max. 1 point).
   (ii) Derive the formula for the Black-Scholes price of call and put options (max. 2 points).
   (iii) Derive the density of the geometric Brownian motion (max. 2 points).

   Solution. See lecture notes. The Black-Scholes price arises as the time-continuum limit of the binomial price.

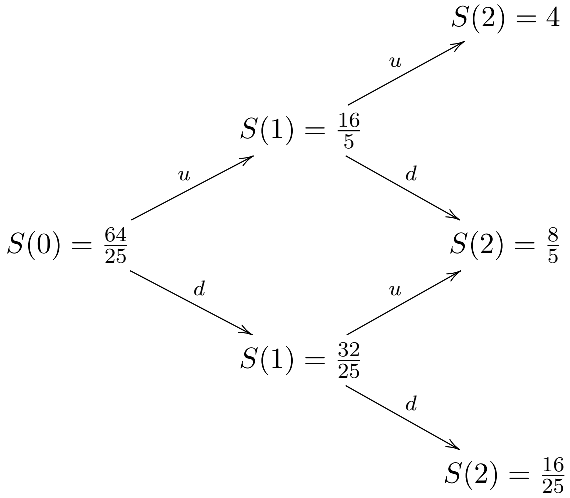
2. Consider a 2-period binomial model with the following parameters:

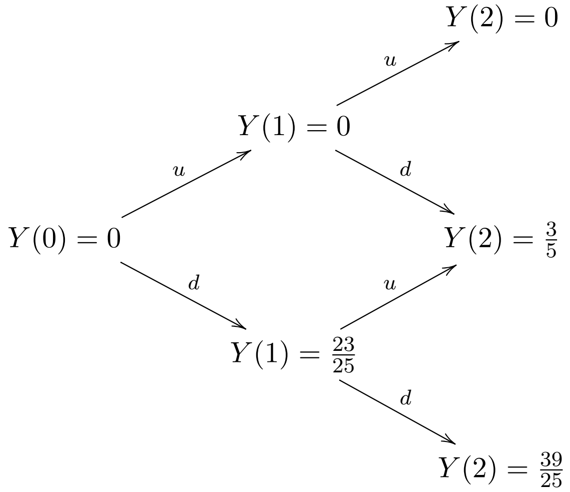
$$e^u = \frac{5}{4}, \quad e^d = \frac{1}{2}, \quad e^r = 1, \quad p \in (0,1).$$

   Let $S(0) = \frac{64}{25}$ be the initial price of the stock and $B(0) = 1$. Compute the price at time $t \in \{0,1,2\}$ of the American put on the stock with maturity $T = 2$ and strike price $K_2 = \frac{11}{5}$ and identify the possible optimal exercise times prior to maturity (max. 1 point). Next consider the compound option which gives to its owner the right to buy the American put at time $t = 1$ for the price $K_1 = \frac{8}{25}$. Compute the price of the compound option at time $t = 0$ (max. 1 point) and the hedging portfolio for the compound option (max. 1 point). Compute the maximum expected return in the interval $t \in [0,2]$ for the owner of the compound option as a function of $p \in (0,1)$ (max. 2 points).

   Solution. The binomial tree for the stock price and for the intrinsic value $Y(t)$ of the

American put are





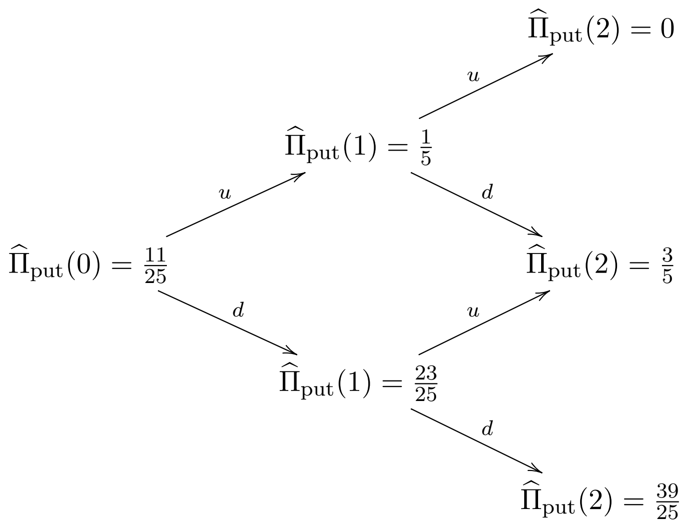
Let $\widehat{\Pi}_{\text{put}}(t)$ be the price at time $t \in \{0, 1, 2\}$ of the American put. We have the recurrence formula $\widehat{\Pi}_{\text{put}}(2) = Y(2)$ and

$$\widehat{\Pi}_{\text{put}}(t) = \max[Y(t), e^{-r}(q_u \widehat{\Pi}^u_{\text{put}}(t) + q_d \widehat{\Pi}^d_{\text{put}}(t))],$$

where

$$q_u = \frac{e^r - e^d}{e^u - e^d} = \frac{2}{3}, \quad q_d = 1 - q_u = \frac{1}{3}, \quad e^r = 1.$$

Hence the binomial tree for $\widehat{\Pi}_{\text{put}}(t)$ is



The only optimal exercise time prior to maturity is $t = 1$ when $S(1) = \frac{32}{25}$. This concludes the first part of the exercise (1 point). The compound option has maturity $T = 1$ and pay-off

$$Q = (\widehat{\Pi}_{\text{put}}(1) - \frac{8}{25})_+.$$

Since $\widehat{\Pi}_{\text{put}}(1)$ is a function of $S(1)$, then we can treat the compound option as a standard derivative on the stock. The compound option expires in the money if the stock price goes down at time $t = 1$ and out of the money otherwise. Hence the price of the compound option at time $t = 0$ is

$$\Pi_{\text{cp}}(0) = \frac{1}{3}(\frac{23}{25} - \frac{8}{25})_+ = \frac{1}{5}.$$

This answers the second question (1 point). As to the hedging portfolio, the compound option can be hedged by investing on the stock and the risk-free asset. The number of shares in the stock is

$$h_S = \frac{1}{S(0)} \frac{\Pi^u_{\text{cp}}(1) - \Pi^d_{\text{cp}}(1)}{e^u - e^d} = \frac{25}{64} \frac{0 - \frac{3}{5}}{\frac{5}{4} - \frac{1}{2}} = -\frac{5}{16}$$

The number of shares in the risk-free asset is obtained by solving the replicating equation at time $t = 0$:

$$h_S S(0) + h_B B(0) = \Pi_{\text{cp}}(0) \Rightarrow h_B = \frac{\Pi_{\text{cp}}(0) - h_S S(0)}{B(0)} = \frac{31}{25}.$$

This answers the third question (1 point). Finally we compute the expected return $\mathbb{E}[R]$ for the owner of the compound option as a function of $p \in (0, 1)$. Clearly

$$R = -\frac{1}{5} \quad \text{with prob. } p,$$

which is the return when the stock price goes up at time $t = 1$. If the stock price goes down at time $t = 1$, the owner of the compound option will buy the American put for $K_1 = 8/25$. If the American put is exercised at this optimal exercise time, then the return will be

$$R = \frac{23}{25} - \frac{1}{5} - \frac{8}{25} = \frac{2}{5} \quad \text{with prob. } 1 - p.$$

Hence, if the American put is exercised at $t = 1$, the expected return is

$$\mathbb{E}[R] = -\frac{1}{5}p + \frac{2}{5}(1-p) = \frac{2}{5} - \frac{3}{5}p.$$

If the American put is exercised at $t = 2$, the expected return is

$$\mathbb{E}[R] = -\frac{1}{5}p + (\frac{3}{5} - \frac{8}{25} - \frac{1}{5})p(1-p) + (\frac{39}{25} - \frac{8}{25} - \frac{1}{5})(1-p)^2 = \frac{1}{25}(3p-2)(8p-13) = f(p).$$

Now, it is straightforward to verify that $f(p) > \frac{2}{5} - \frac{3}{5}p$ when $0 < p < 2/3$ and $f(p) < \frac{2}{5} - \frac{3}{5}p$ when $2/3 < p < 1$. Hence the strategy which maximizes the expected return for the compound option is: for $0 < p < 2/3$, the American put should *not* be exercised at time $t = 1$, while for $2/3 < p < 1$ the American put should be exercised at time $t = 1$. For $p = 2/3$ the two strategies lead to the same expected return. This answers the last question (2 points).

3. (Put-call parity for Asian options). Consider a N-period arbitrage-free binomial market with $r \neq 0$ and let $S(t)$ denote the price of the stock at time $t \in \{0, \ldots, N\}$. The Asian call, resp. put, with maturity $T = N$ and strike K is the non-standard European style derivative with pay-off

$$Y_{\text{call}} = \left[\Big(\frac{1}{N+1}\sum_{t=0}^{N} S(t)\Big) - K\right]_+, \quad \text{resp.} \quad Y_{\text{put}} = \left[K - \Big(\frac{1}{N+1}\sum_{t=0}^{N} S(t)\Big)\right]_+$$

Denote by $AC(0)$ and $AP(0)$ the binomial price at time $t = 0$ of the Asian call and put, respectively. Prove the following put-call parity identity:

$$AC(0) - AP(0) = e^{-rN}\Big[\frac{S(0)}{N+1}\frac{e^{r(N+1)} - 1}{e^r - 1} - K\Big] \quad \text{(max. 5 points).}$$

Solution. We have

$$AC(0) = e^{-rN} \sum_{x \in \{u,d\}^N} (q_u)^{N_u(x)}(q_d)^{N_d(x)} Y_{\text{call}}(x)$$

and similarly for the Asian put. Thus

$$AC(0) - AP(0) = e^{-rN} \sum_{x \in \{u,d\}^N} (q_u)^{N_u(x)}(q_d)^{N_d(x)} (Y_{\text{call}}(x) - Y_{\text{put}}(x)).$$

Using

$$Y_{\text{call}} - Y_{\text{put}} = \left[\Big(\frac{1}{N+1}\sum_{t=0}^{N} S(t)\Big) - K\right]_+ - \left[K - \Big(\frac{1}{N+1}\sum_{t=0}^{N} S(t)\Big)\right]_+ = \Big(\frac{1}{N+1}\sum_{t=0}^{N} S(t)\Big) - K$$

we find

$$\begin{aligned} AC(0) - AP(0) &= e^{-rN} \sum_{x \in \{u,d\}^N} (q_u)^{N_u(x)}(q_d)^{N_d(x)} \frac{1}{N+1}\sum_{t=0}^{N} S(t) \\ &\quad - e^{-rN} K \sum_{x \in \{u,d\}^N} (q_u)^{N_u(x)}(q_d)^{N_d(x)} \\ &= \frac{e^{-rN}}{N+1}\left(\sum_{t=0}^{N} \mathbb{E}[S(t)]\right) - K e^{-rN}\mathbb{E}_q[1], \end{aligned}$$

where $\mathbb{E}_q[\cdot]$ denotes the expectation in the risk-neutral measure. As $\mathbb{E}_q[1] = 1$ and $\mathbb{E}_q[S(t)] = S(0)e^{rt}$, we obtain

$$AC(0) - AP(0) = e^{-rN}\Big[\frac{S(0)}{N+1}\left(\sum_{t=0}^{N} e^{rt}\right) - K\Big].$$

Using the formula in the HINT concludes the exercise (5 points).