MATEMATIK	Datum: 2014-08-25	Tid: eftermiddag
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Lösningar till tenta i ODE och matematisk modellering, MMG511, MVE161

1. Define what is a generalized eigenspace of a matrix. Formulate the theorem on the stability of solutions to a linear system of ODEs $\frac{d\overrightarrow{r}(t)}{dt} = A\overrightarrow{r}(t)$ with a constant $n \times n$ matrix A using generalized egenspaces of the matrix A. Describe main ideas of the proof. **(4p)**

   Let v_s be an eigenvector corresponding to a possibly multiple eigenvalue λ_j, then the generalized eigenvector $v_s^{(1)}$ is a solution of the equation $(\lambda_j I - A)\, v_s^{(1)} = v_s$ and more generalized eigenvectors $v_s^{(k)}$associated with v_s are found as nontrivial solutions to the equations $(\lambda_j I - A)\, v_s^{(k)} = v_s^{(k-1)}$ for $k = 1, ..., l_s$ if they exist. It is easy to observe that $v_s^{(k)}$ satisfies the equation $(\lambda_j I - A)^{k+1}\, v_s^{(k)} = 0$.

   Generalized eigenspace $M(\lambda_j, A)$ is a subspace of all eigenvectors and generalized eigenvectors corresponding to the eigenvalue λ_j. Each generalized eigenspace $M(\lambda_j, A)$ can in principle include several linearly independent eigenvectors and several (or no one) linearly independent generalized eigenvectors.

   Theorem 3.3.5, p. 56 in the book by Hsu.

   Two statements are formulated there:

   a) if all eigenvalues λ_j of the matrix A have strictly negative real part $\operatorname{Re}\lambda_j < 0$, then there are constants $0 < K$ and $0 < \alpha$ such that $\left\|e^{At}\right\| \leq Ke^{-\alpha t}$ for all $t \geq 0$.

   b) if a weaker estimate is valid $\operatorname{Re}\lambda_j \leq 0$ and those egenvalues that have zero real part have the same algebraic and geometric multyplicity (or have simple elementary divisors) then there is a constant $M > 0$ such that $\left\|e^{At}\right\| \leq M$.

   Proof is based on the representation $e^{At} = Pe^{Jt}P^{-1}$ of e^{At}, where matrix P consists of columns that are include a maximal possible set of eigenvectors v_k followed by corresponding generalised eigenvectors $v_k^{(i)}$. These eigenvectors together with generalized eigenvectors build a basis in $\mathbb{R}^n$. J is a block diagonal matrix with square Jordan blocks J_k corresponding to linearly independent eigenvectors of all distinct eigenvalues of A. Blocks follow the order in which eigenvectors and corresponding eigenvalues stand in P. Each eigenvalue λ_j with multiplicitet m can have from one to m linearly independent eigenvectors. The last case means that the eigenvalue has the same algebraic multiplicity and geometric multyplicity. In the last case all Jordan blocks corresponding to λ_j will consist of 1×1 matrix and will build up a diagonal block in J.

   The rool $\|BC\| \leq \|B\|\,\|C\|$ for the norm of matrix product implies that

   $\left\|e^{At}\right\| \leq \|P\|\,\left\|e^{Jt}\right\|\,\left\|P^{-1}\right\|$ where norms $\|P\|$ and $\left\|P^{-1}\right\|$ are both bounded. Because of the block-diagonal structure of e^{Jt} following from the block - diagonal structure of J the following estimate is valid:$\left\|e^{Jt}\right\| \leq \max_k \left\|e^{J_k t}\right\|$ and the estimating of $\left\|e^{At}\right\|$ is reduced to the estimating of all $\left\|e^{J_k t}\right\|$.

   $e^{J_k t} = e^{\lambda_k t}e^{N_k t}$, where $e^{N_k t}$ in case it has dimension larger than 1, includes powers of t^m, with $m < n$.

   N_k is the matrix with all elements zero exept those over the main diagonal that are ones.

   If $\min(\operatorname{Re}\lambda_k) < 0$, and we choose an $\alpha < |\min(\operatorname{Re}\lambda_k)|$ then product of $e^{\lambda_k t}t^m < Ke^{-\alpha t}$

for all $m \le n$ and some constant K, because any any power m, and any $\varepsilon > 0$, $t^m e^{-\varepsilon t}$ is bounded for all $t \ge 0$. It implies the first statement.

The second statement follows from the same reasoning as above and additional proof for those λ_k that have $\operatorname{Re}\lambda_k = 0$. We observe that if λ_k has simple divisors (the same algebraic and geometric multyplicity), then N_k above will have dimension 1 and there will be no powers of t in the expression for $e^{\lambda_k N_k t}$. If N_k has dimension one, then it is just zero and $e^{N_k t} = 1$. Exponentials $e^{\lambda_k t}$ will be bounded for $\operatorname{Re}\lambda_k = 0, \left|e^{\lambda_k t}\right| \le 1$, because $e^{\lambda_k t} = \cos(\operatorname{Im}\lambda_k t) + i\sin(\operatorname{Im}\lambda_k t)$.

2. Formulate and prove the theorem on stability and asymptotical stability of equilibrium points to autonomous ODEs by Lyapunovs functions. **(4p)**

   See Theorem 5.2.1, p. 134-135 in the book by Hsu.

3. Consider the following system of ODE:

   $$\frac{d\overrightarrow{r}(t)}{dt} = A\overrightarrow{r}(t), \text{ with a constant matrix } A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 4 \\ 1 & 0 & -1 \end{bmatrix}.$$

   Find general solution to the system. Find all those initial vectors $\overrightarrow{r_0} = \overrightarrow{r}(0)$ that give bounded solutions to the system. **(4p)**

   Eigenvectors and eigenvalues are : $v_1 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \leftrightarrow \lambda_1 = 0, v_2 = \begin{bmatrix} 4 \\ 4 \\ 1 \end{bmatrix} \leftrightarrow \lambda_2 = 3$

   Generalized eigenvector $v_1^{(1)}$ corresponding to λ_1 satisfies the equation $Av_1^{(1)} = v_1$. (there are infinitely many possible solutions)

   $v_1^{(1)} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$. Jordan form of the matrix is $J = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

   $A = PJP^{-1}$ with $P = \begin{bmatrix} 1 & 0 & 4 \\ -2 & 1 & 4 \\ 1 & -1 & 1 \end{bmatrix}$.

   General solution can be found in the general form:

   $$x(t) = e^{At}x_0 = \sum_{j=1}^{s}\left(\left[\sum_{k=0}^{n_j-1}(A-\lambda_j I)^k \frac{t^k}{k!}\right] x^{0,j} e^{\lambda_j t}\right)$$

   $x_0 = \sum_{j=1}^{s} x^{0,j}$ where $x^{0,j} \in M(\lambda_j, A)$ - are projections of x_0 to all generalized eigenspaces $M(\lambda_j, A)$ of the matrix A corresponding to each eigenvector.

   We choose x_0 in the form $x_0 = C_1 v_j + C_2 v_1^{(1)} + C_3 v_2$ where $C_1 v_j + C_2 v_1^{(1)} = x^{0,1} \in M(\lambda_1, A)$ and v_j and $v_1^{(1)}$ constitute basis for $M(\lambda_1, A)$. $M(\lambda_2, A)$ has dimension one because the eigenvalue λ_2 is simple and $x^{0,2} = C_3 v_2$ in the general formula above.

   $x(t) = e^{At}x_0 = e^{\lambda_1 t}\left(C_1 v_j + C_2 v_1^{(1)}\right) + te^{\lambda_1 t}(A - \lambda_j I)\left(C_1 v_j + C_2 v_1^{(1)}\right) + C_3 e^{\lambda_2 t} v_2 =$

   $e^{\lambda_1 t}\left(C_1 v_1 + C_2 v_1^{(1)}\right) + te^{\lambda_1 t}(C_2 v_1) + C_3 e^{\lambda_2 t} v_2$

   $$x(t) = C_1\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} + C_2\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + tC_2\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} + C_3 e^{3t}\begin{bmatrix} 4 \\ 4 \\ 1 \end{bmatrix} =$$

$$\begin{bmatrix} C_1 + tC_2 + 4C_3 e^{3t} \\ -2C_1 + C_2 - 2tC_2 + 4C_3 e^{3t} \\ C_1 - C_2 + tC_2 + C_3 e^{3t} \end{bmatrix} = \begin{bmatrix} C_1 + tC_2 + 4C_3 e^{3t} \\ -2C_1 + (1-2t)\,C_2 + 4C_3 e^{3t} \\ C_1 + (-1+t)\,C_2 + C_3 e^{3t} \end{bmatrix}$$

You can get the answer in infinitely many different forms depending on the choice of eigenvectors and generalized eigenvectors as a base for initial data in $\mathbb{R}^3$.

The solution will be bounded only if initial data are parallel to v_1, or if $C_2 = C_3 = 0$ in our representation.

4. Consider the following initial value problem: $y' = y^2 + t$; $y(1) = 0$.

a) Reduce the problem to an integral equation. Calculate three first Picard approximations as in the proof to Picard Lindelöf theorem.

b) Find some time interval where the Picard approximations converge.

(4p)

a) The IVP for an equation $y' = f(y,t),\quad y(t_0) = y_0$ can be rewritten in integral form: $y(t) = y(t_0) + \int_{t_0}^{t} f(y(s), s)ds$.

Picard iterations are $y_n(t) = y(t_0) + \int_{t_0}^{t} f(y_{n-1}(s), s)ds$; with $y_0(t) = y(t_0)$.

In our particular case $t_0 = 1,\ \ y(1) = 0$, $y(t) = \int_1^t \left(y^2(s) + s\right) ds$. Three Picard iterations for the given example are:

$y_0(t) = 0;\ \ y_1(t) = \int_1^t (s)\, ds = \frac{t^2-1}{2}$;

$y_2(t) = \int_1^t \left(\left(\frac{s^2-1}{2}\right)^2 + s \right) ds = \frac{1}{4}t + \frac{1}{2}t^2 - \frac{1}{6}t^3 + \frac{1}{20}t^5 - \frac{19}{30}$.

b) The Picard Lindelöf theorem states that the IVP for an ODE with the right hand side $f(y,t)$ continuous in the domain $|t - t_0| < a$, $|y - y_0| < b$ such that $|f(y,t)| < M$ for these t and y and satisfying Lipshitz condition with respect to y with Lipshitz constant L has a unique solution on the interval $|t - t_0| < \alpha$ for $\alpha = \min(a, b/M, 1/L)$.

Answering the question reduces to choosing some a and b, finding an estimate M for the particular right hand side and finding an estimate for $\left|\frac{\partial}{\partial y} f(y,t)\right|$ that will give an estimate for the Lipschitz constant $L \le \max\limits_{|t-t_0|<a|y-y_0|<b} \left|\frac{\partial}{\partial y} f(y,t)\right|$.

Then the time interval is given by the formula $|t - t_0| < \alpha$ with $\alpha = \min(a, b/M, 1/L)$.

Choose $a = b = 1$ and point out that $t_0 = 1$. Then $\left|\left(y^2(t) + t\right)\right| \le 3$ for $|t - 1| < 1$; (or $0 < t < 2$) and $|y - 0| < 1$. We choose $M = 3$.

$\frac{\partial}{\partial y} f(y,t) = 2y$, $|2y| \le 2$ for $|y - 0| < 1$. We choose $L = 2$.

then the size of the time interval where iterations converge can be estimated as

$\alpha = \min(a, b/M, 1/L) = \min(1, 1/3, 1/2) = 1/3$.

Answer. The Picard iterations for the given example converge for the time interval $|t - 1| \le 1/3$ (at least).

5. For the following system of equations find all equilibrium points and investigate their stability.

$$\begin{cases} x' = & \ln(-x + y^2) \\ y' = & x - y - 1 \end{cases}$$

(4p)

Jacobian of the right hand side is $J(x,y) = \begin{bmatrix} -\frac{1}{-x+y^2} & 2\frac{y}{-x+y^2} \\ 1 & -1 \end{bmatrix}$

There are two stationary points: (0,-1) and (3,2).

$J(0,-1) = \begin{bmatrix} -1 & -2 \\ 1 & -1 \end{bmatrix}$, eigenvalues: $\lambda_1 = -i\sqrt{2} - 1, \lambda_2 = i\sqrt{2} - 1$.The stationary point $(0,-1)$ is stable because for both eigenvalues $\operatorname{Re}\lambda_i < 0$, it is a stable spiral.

$J(3,2) = \begin{bmatrix} -1 & 4 \\ 1 & -1 \end{bmatrix}$, eigenvalues: $\lambda_1 = -3, \lambda_2 = 1$.The stationary point $(3,2)$ is unstable because one of the eigenvalues $\operatorname{Re}\lambda_2 > 0$, it is a saddle point.

Answer: (0,-1) - stable (3,2) - unstable.

6. Formulate the Poincare-Bendixson theorem and use it to show that the following system of ODE has a periodic solution $(x(t), y(t)) \neq (0,0)$.

$$\begin{cases} x' = & y \\ y' = & -f(x,y)\,y - x \end{cases}$$

where f, f'_x , f'_y are continuous, $f(0,0) < 0$ and $f(x,y) > 0$ for $x^2 + y^2 > b^2$.

(4p)

One of formulations of the Poincare Bendixson theorem is: let $\varphi(t)$ be a solution to an autonomous equation $x' = f(x)$ in plane, bounded for all $t > 0$. We suppose that f is Lipschitz to guarantee uniquness of solutions. Then the limit set $\omega(\varphi)$ of $\varphi(t)$ has the following property: it either

i) contains an equilibrium

or

ii) $\varphi(t)$ is periodic itself

or $\omega(\varphi)$ is a periodic orbit.

The theorem cas an important corollary, that is also often called Poincare Bendixson theorem:

If the equation $x' = f(x)$ in plane has a compact positively invariant set (a set that no trajectories can leave) that does not include any equilibrium points, then this set must include at least one periodic orbit.

We apply the corollary to the example above and try to find a set in plane satisfying requirements i the corollary.

We observe first that solutions exist for initial points everywhere in the plane because of the smoothness of f.

Consider a test function $V(x,y) = \left(x^2 + y^2\right)/2$ and its evolution along solutions to the given system.

$V' = xy - y^2 f(x,y) - xy = -y^2 f(x,y)$. $f(x,y) > 0$ for $x^2 + y^2 > b^2$. It implies that $V' \leq 0$ for $x^2 + y^2 = b^2$ and that trajectories of the system cannot leave the disc $x^2 + y^2 \leq b^2$.

$f(0,0) < 0$ and is continuous. It implies that there is a small circle around the origin $x^2 + y^2 \leq \delta^2$

such that $f(x,y) < 0$ and correspondingly $V' > 0$ for $x^2 + y^2 = \delta^2$. It implies that the system cannot leave the ring $\delta^2 \leq x^2 + y^2 \leq b^2$. It is easy to observe that the only equilibrium point of the system is the origin: $y = 0$ for the first equation and therefore $x = 0$ from the second equation.

Alltogether implies that the ring $\delta^2 \leq x^2 + y^2 \leq b^2$ is a positively invariant set without equilibrium points and must include at least one periodic orbit.

Max: 24 points;

Thresholding for marks: for GU: **VG:** 19 points; **G:** 12 points. For Chalmers: **5:** 21 points; **4:** 17 points; **3**: 12 points;

One must pass both the home assignments and the exam to pass the course.

Total points for the course will be the average of the points for the home assignments (30%) and for this exam (70%).

The same thresholding is valid for the exam, for the home assignments, and for the total points.