MATEMATIK	Datum: 2015-06-01	Tid: 8:30
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Lösningar till tenta i ODE och matematisk modellering, MMG511, MVE161

Answer first those questions that look simpler, then take more complicated ones etc.
Good luck!

1. Formulate and give a proof to the theorem about the limit and boundedness of solutions $\overrightarrow{x}(t)$ to linear systems of ODE with constant coefficients when $t \to +\infty$. **(4p)**

   Check Theorem 3.3.5 in the course textbook.

2. Formulate and give a proof to Bendixsons criteria about non-existence of periodic solutions to autonomous ODE in plane. **(4p)**

   Check Theorem 6.1.2 in the course textbook.

3. Consider the following system of ODE:

$$\frac{d\overrightarrow{r}(t)}{dt} = A\overrightarrow{r}(t), \text{ with a constant matrix } A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & -2 & 1 \end{bmatrix}.$$

Give a general real solution to the system. Find all those initial vectors $\overrightarrow{r_0} = \overrightarrow{r}(0)$ that give bounded solutions to the system. **(4p)**

Solution. The general solution has the form $\overrightarrow{r}(t) = \exp(tA)\overrightarrow{r_0}$ for $\overrightarrow{r}(0) = \overrightarrow{r_0}$. The matrix A is block-diagonal, so the matrix exponential has the form

$\exp(tA) = \begin{bmatrix} \exp(tJ) & \mathbb{O} \\ \mathbb{O} & \exp(tK) \end{bmatrix}$ with $J = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $K = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$, and $\mathbb{O}$ - zero 2×2 matrix.

We know that $\exp(tJ) = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$ and $\exp(tK) = \exp(t) \begin{bmatrix} \cos(2t) & \sin(2t) \\ -\sin(2t) & \cos(2t) \end{bmatrix}$.

The first relation follows from the power series for exponent $\exp(tJ)$, that for the matrix J will consist just of two nonzero terms, because $J^2 = 0$. The second relation follows from the fact that the matrices in the form: $\mathbb{K} = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$ have the same algebraic properties as complex numbers $a + ib$ and from the Euler formula for the exponent of complex numbers: $\exp(a + ib) = \exp(a)(\cos(b) + i\sin(b))$.

If one does not remember the formula for $\exp\left(\begin{bmatrix} a & b \\ -b & a \end{bmatrix}\right)$, then one can easily derive it by solving the system $\overrightarrow{w}' = \mathbb{K}\overrightarrow{w}$.

We observe that the arbitrary matrix in the form $\mathbb{K} = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$, has complex conjugate eigenvectors and eigenvalues : $v_2 = \left\{\begin{bmatrix} -1 \\ i \end{bmatrix}\right\} \leftrightarrow \lambda_2 = a - ib$, and $v_1 = \left\{\begin{bmatrix} 1 \\ i \end{bmatrix}\right\} \leftrightarrow \lambda_1 = a + ib$. The general complex solution to the system $\overrightarrow{w}' = \mathbb{K}\overrightarrow{w}$ has the form: $\overrightarrow{w}(t) = C_1 \exp(\lambda_1 t)v_1 + C_2 \exp(\lambda_2 t)v_2$.

We can instead choose real and imaginary parts of $\exp(\lambda_1 t)v_1$ as a basis for real solutions. We use here the Euler formula for the exponent of complex numbers:

$$V_1(t) = \mathrm{Re}\left(\exp\left((a+bi)t\right)\begin{bmatrix} 1 \\ i \end{bmatrix}\right) = \begin{bmatrix} e^{at}(\cos bt) \\ -e^{at}(\sin bt) \end{bmatrix};$$

$$V_2(t) = \mathrm{Im}\left(\exp\left((a+bi)t\right)\begin{bmatrix} 1 \\ i \end{bmatrix}\right) = \begin{bmatrix} e^{at}(\sin bt) \\ e^{at}(\cos bt) \end{bmatrix}$$

We observe that $V_1(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $V_2(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. It implies that the fundamental matrix solution $[V_1(t), V_2(t)]$ is the principal matrix solution: $[V_1(0), V_2(0)] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$ and $[V_1(t), V_2(t)] = \exp(t\mathbb{K}) = \exp(at)\begin{bmatrix} \cos(bt) & \sin(bt) \\ -\sin(bt) & \cos(bt) \end{bmatrix}$. In particular we get the formula for the exponent of the matrix tK above .

Finaly the general real solution to the given system is

$$\overrightarrow{r}(t) = \exp(tA)\overrightarrow{r_0} = \begin{bmatrix} 1 & t & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & e^t(\cos 2t) & e^t(\sin 2t) \\ 0 & 0 & -e^t(\sin 2t) & e^t(\cos 2t) \end{bmatrix}\overrightarrow{r_0} \text{ for an arbitrary initial vector } \overrightarrow{r_0}.$$

It is easy to observe that the only initial data that give bounded solutions consist of vectors $\overrightarrow{r_0} \in \mathbb{R}^4$ with all components exept the first one equal to zero, and of the zero initial data, because all columns in $\exp(tA)$ exept the first one include unbounded functions. The last question can be answered even using the simpler complex form of the general solution: $\overrightarrow{r}(t) = \exp(tA)\overrightarrow{r_0} = \begin{bmatrix} 1 & t & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & e^{\lambda_1 t} & 0 \\ 0 & 0 & 0 & e^{\lambda_2 t} \end{bmatrix}\overrightarrow{r_0}$ with $\lambda_1 = 2+i$, $\lambda_2 = 2-i$ - being complex eigenvalues to the matrix $K = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$.

4. Formulate Banach's contraction principle. Consider the following operator

$$K(x)(t) = A\int_0^\pi \sin(ts)\, x(s)ds + t^2,$$

for all $t \in [0,\pi]$ acting in the Banach space $C([0,\pi])$ of continuous functions with norm $\|x\| = \sup_{t\in[0,\pi]} |x(t)|$.

Find using Banach's contraction principle conditions on the constant $A > 0$ such that the operator $K(x)(t)$ has a fixed point. **(4p)**

Solution. Banachs contraction principle states that if an operator K maps a closed subset U in a Bancha space B into itself: $K : U \to U$

and is a contraction on U, that means that $\|Kx - Ky\| \le \theta \|x - y\|$ with $\theta < 1$, then the operator K has a unique fixed point $\widetilde{x} = K\widetilde{x}$ in U that can be found by iterations

$x_{n+1} = Kx_n$ with an arbitrary start approximation $x_0 \in U$, so that $x_{n+1} \underset{n\to\infty}{\longrightarrow} \widetilde{x}$.■

We calculate supremum norm of the value of the operator $K(x)(t)$:

$\|Kx\| = \sup_{t\in[0,\pi]} |K(x)(t)| = \sup_{t\in[0,\pi]} \left|A\int_0^\pi \sin(ts)\, x(s)ds + t^2\right| \le \pi^2 + A\pi \sup_{s\in[0,\pi]} |x(s)|$

Therefore $\|Kx\| \le \pi^2 + A\pi \|x\|$. It implies that the operator K maps a ball with radius r in the Banach space $C([0,\pi])$ into the ball of radius $\pi^2 + A\pi r$.

We like to find such radius R of the ball $B(0,R)$ in $C([0,\pi])$ and such constant $A>0$ that K would map the ball $B(0,R)$ into itself. Namely that $\pi^2+A\pi R\leq R$. It implies $\pi^2\leq R(1-A\pi)$. We see that A must be chosen smaller than $1/\pi$: $A<\frac{1}{\pi}$ and R must be chosen large enough: $\frac{\pi}{(1-A\pi)}\leq R$. Then $K:B(0,R)\to B(0,R)$. The next step is to find conditions that imply that K is a contraction on $B(0,R)$. We estimate the norm $\|Kx-Ky\|$:

$$\|Kx-Ky\|\leq\sup_{t\in[0,\pi]}\left|A\int_0^\pi\sin(ts)\ (x(s)-y(s))\,ds\right|\leq A\pi\sup_{s\in[0,\pi]}|x(s)-y(s)|=A\pi\,\|x-y\|$$

We have chosen already $A<\frac{1}{\pi}$. It implies that K is a contraction on $B(0,R)$ with $\frac{\pi}{(1-A\pi)}\leq R$ chosen above so that $K:B(0,R)\to B(0,R)$. It implies by the Banach contraction principle that K has a unique fixed point in $B(0,R)$.

5. Consider the following system of ODE and investigate the stability of the stationary point in the origin depending on the real constant $a\in\mathbb{R}$.

$$\begin{cases}x'=y\\ y'=-x+(a-x^2)y\end{cases}\qquad\textbf{(4p)}$$

Solution. We try to use the linearization of the system. The variational matrix in the origin is $A=\begin{bmatrix}0&1\\-1&a\end{bmatrix}$.

Eigenvalues of A are $\lambda_1=\frac{1}{2}a-\frac{1}{2}\sqrt{a^2-4},\lambda_2=\frac{1}{2}a+\frac{1}{2}\sqrt{a^2-4}$. For $a>0$ we see that $\mathrm{Re}(\lambda_i)>0$ and the Grobman-Hartman theorem implies that the origin is unstable (even repeller). Similarly for $a<0$, $\mathrm{Re}(\lambda_i)<0$ and the origin is asymptotically stable. For $0<a<2$ it will be an unstable spiral, for $2<a$ it will be an unstable node. For $a=2$ it will be an unstable improper node. For $-2<a<0$ it will be stable spiral, for $a<-2$ it will be a stable node. For $a=-2$ it will be stable improper node.

For For $a=0$ we cannot use Grobman Hartman theorem because the origin is not hyperbolic: both eigenvalues have real part zero.

We try instead to use a simple test function $V(x,y)=\frac{1}{2}\left(x^2+y^2\right)$ in this case. Introducing the vector notation $f(x,y)$ for the right hand side of the equation we get

$V'(x,y)=\nabla V\cdot f(x,y)=\begin{bmatrix}x\\y\end{bmatrix}\cdot\begin{bmatrix}y\\-x-x^2y\end{bmatrix}=xy-yx-y(x)^2y=-x^2y^2\leq 0$ for $(x,y)\neq(0,0)$. Therefore the origin is a stable stationary point. We use Lasalle's invariance principle to check if the origin is asymptotically stable or not. $V'(x,y)=0$ on the set S where x=0 or $y=0$: the union of coordinate axises. We check if this set includes invariant sets other than the origin. For $x=0$, $x'=y\neq 0$ for $y\neq 0$. For $y=0$, $y'=-x\neq 0$ for $x\neq 0$. Therefore the set S includes only one invariant set - the origin $(0,0)$, that by a corollary to Lasalle's invariance principle must be asymptotically stable.

6. Show that all solutions to the following system of ODE exist for arbitrary large time $t>0$

$$\begin{cases}x'=-4x^3+2xy\\ y'=-2y+x^2\end{cases}\qquad\textbf{(4p)}$$

Solution. We try to show that all solutions stay within a finite domain. It would imply that they all are extendable for any time $t>0$. We use a simple test function $V(x,y)=\frac{1}{2}x^2+y^2$ in this case. Introducing vector notations for the right hand side $f(x,y)$ of the equation we get

$V'(x,y)=\nabla V\cdot f(x,y)=\begin{bmatrix}x\\2y\end{bmatrix}\cdot\begin{bmatrix}-4x^3+2xy\\-2y+x^2\end{bmatrix}=2x^2y-4y^2-4x^4+2x^2y=-4\left(x^4-x^2y+y^2\right)<0$ for $(x,y)\neq(0,0)$

because the quadratic form $a^2 - ab + b^2$ is positive definite.

It means that solutions starting inside an ellipse $\frac{1}{2}x^2 + y^2 < C$ of radius $C > 0$ will never leave it and therefore can be extended for any time $t > 0$ because the right hand side of the equation is a smooth function in the whole plane $\mathbb{R}^2$ (and therefore is Lipschitz in any bounded domain).

Max. 24 points;

Thresholding for marks: for GU: **VG:** 19 points; **G:** 12 points. For Chalmers: **5:** 21 points; **4:** 17 points; **3**: 12 points;

One must pass both the home assignments and the exam to pass the course. Total points for the course are calculated as $Total = 0.3 Assignments + 0.7 Exam$ - the average of the points for the home assignments (30%) and for this exam (70%). The same thresholding is valid for the exam, for the home assignments, and for the total points for the course.