MATEMATIK	Datum: 2016-08-22	Tid: 14-00 - 18-00
GU, Chalmers	Hjälpmedel: - Inga	
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Lösningar till tenta i ODE och matematisk modellering, MMG511, MVE162 (MVE161)

Answer first those questions that look simpler, then take more complicated ones etc.
Good luck!

1. Give the definition for a stable equilibrium point to an autonomous system of ODEs. Formulate and prove the theorem on the stability of equilibrium points to autonomous ODEs by Lyapunovs functions. **(4p)**

2. Formulate and give a proof to Bendixsons criteria about non-existence of periodic solutions to autonomous ODE in the plane. **(4p)**

   Check proofs in the textbook.

3. Consider the following system of ODE: $\dfrac{d\overrightarrow{r}(t)}{dt} = A\overrightarrow{r}(t)$, with a constant matrix $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 4 \\ 1 & 0 & -1 \end{bmatrix}$. Give general solution to the system. Find all initial data such that corresponding solutions are bounded. **(4p)**

   Solution. General solution is can be given in two ways: as $\overrightarrow{r}(t) = \exp(tA)\overrightarrow{r}_0$ for arbitrary vector $\overrightarrow{r}_0$ of initial data at $t = 0$, alternativly as a linear combination of columns in an arbitrary fundamental matrix solution $\Phi(t)$.

   Practical calculation of $\exp(tA)\overrightarrow{r_0}$ is based on the idea of considering $\exp(tA)\overrightarrow{r}_0 = e^{\lambda_j t} e^{t(A-\lambda_j I)}\overrightarrow{r}_0$ for an eigenvalue λ_j with multiplicity n_j.

   It is easy to observe that for initial data $\overrightarrow{r}_0$ represented as a linear combination of the eigenvectors and generalized eigenvectors to A corresponding to the eigenvalue λ_j or $\overrightarrow{r}_0 \in M(\lambda_j, A)$, the general expression $\exp(tA)\overrightarrow{r}_0 = e^{\lambda_j t}\sum_{k=0}^{\infty}\frac{t^k}{k!}(A-\lambda_j I)^k\overrightarrow{r}_0$ becomes a finite sum:

   $$\exp(tA)\overrightarrow{r}_0 = e^{\lambda_j t}\sum_{k=0}^{n_j-1}\frac{t^k}{k!}(A-\lambda_j I)^k\overrightarrow{r}_0$$

   The set of all eigenvectors and generalized eigenvectors to A builds a basis to the space $\mathbb{C}^n$ and therefore the general solution $\exp(tA)\overrightarrow{r}_0$ can be explicitely expressed as a linear combination of expressions as above for all distinct eigenvalues of A. We remind that generalised eigenvalues $v^{j,(k)}$ satisfy the following equations $(A-\lambda_j I)\, v^{j,(k)} = v^{j,(k-1)}$ for $k > 1$, and $(A-\lambda_j I)\, v^{j,(1)} = v^j$ where v^j is a eigenvector corresponding to λ_j and $v^{j,(k)}$ are associated generalized eigenvectors.

   For the given matrix A, the characteristic polynomial is $\lambda^3 - 3\lambda^2 = 0$ the characteristic values are $\lambda_1 = 0$ (with multiplicity 2) and $\lambda_2 = 3$ (simple). Eigenvectors v^1 and v^2 satisfy equations $(A-0I)\, v^1 = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 4 \\ 1 & 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$, and $(A-3I)v^2 = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 4 \\ 1 & 0 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$,or equivalently $\begin{cases} 2x+y=0 \\ 2y+4z=0 \\ x-z=0 \end{cases}$, with solution $[x = z, y = -2z]$, and $\begin{cases} -x+y=0 \\ -y+4z=0 \\ x-4z=0 \end{cases}$, with solution: $[x = 4z, y = 4z]$. We see that the eigenvalue $\lambda_1 = 0$ has only a one-dimensional set of eigenvectors. We specify one eigenvector by choosing $z = 1$: $v^1 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ and the eigenvalue $\lambda_2 = 3$ has also only one eigenvector (this we could expect because this eigenvalue is simple). A corresponding eigenvector can be chosen as $v^2 = \begin{bmatrix} 4 \\ 4 \\ 1 \end{bmatrix}$. We calculate a generalized eigenvector $v^{1,(1)}$ corresponding to $\lambda_1 = 0$ by solving the equation

$$\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 4 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \text{ or } \begin{cases} 2x + y = 1 \\ 2y + 4z = -2 \\ x - z = 1 \end{cases}, \text{ Solution is: } [x = z + 1, y = -2z - 1]$$

We choose a generalized eigenvector as $v^{1,(1)} = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$ taking $z = 1$.

Representing arbitrary initial vector $\overrightarrow{r}_0 = C_1 v^1 + C_2 v^{1,(1)} + C_3 v^2$ where $C_1 v^1 + C_2 v^{1,(1)} \in M(0, A)$ and $C_3 v^2 \in M(3, A)$, represent general solution as

$$\begin{aligned} \overrightarrow{r}(t) &= \sum_{k=0}^{1} \frac{t^k}{k!} (A - 0I)^k \left(C_1 v^1 + C_2 v^{1,(1)} \right) + C_3 e^{3t} v^2 = C_1 v^1 + C_2 \left(v^{1,(1)} + t v^1 \right) + C_3 e^{3t} v^2 = \\ & C_1 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} + C_2 \begin{bmatrix} 2 + t \\ -3 - 2t \\ 1 + t \end{bmatrix} + C_3 e^{3t} \begin{bmatrix} 4 \\ 4 \\ 1 \end{bmatrix}. \end{aligned}$$

It is easy to observe that solutions are bounded if and only if $r_0 = C_1 v^1$.

4. Consider the system of ODEs. $\begin{cases} x' = y \\ y' = -y - g(x) \end{cases}$,

where g is continuously differentiable for $|x| < k$, for some $k > 0$, and $xg(x) > 0$ for $x \neq 0$, $g(0) = 0$.

Investigate stability of the stationary point at the origin by using an appropriate Lyapunov function. **(4p)**

Solution. Introduce a positive definite test function $V(x, y) = \frac{1}{2} y^2 + \int_0^x g(\lambda) d\lambda$. The derivative of $V(x, y)$ along trajectories of solutions is: $\frac{d}{dt} V(x(t), y(t)) = \begin{bmatrix} g(x) \\ y \end{bmatrix} \cdot \begin{bmatrix} y \\ -y - g(x) \end{bmatrix} = -y^2 \leq 0$

for $(x, y) \neq (0, 0)$. Therefore the origin is a stable stationary point. The set of zeroes of $\frac{d}{dt} V(x(t), y(t))$ consists of the x- axis. The second eqution in the system together with the properties of g - function imply that the origin $(0, 0)$ is the only invariant set for the system on the x - axis. Therefore the origin is an asymptotically stable stationary point by a corollary to the Krasovsky - la Salles invariance principle.

5. Show that the following system of ODEs has a periodic solution.

$$\begin{cases} x' = y - x \left(x^2 + y^2 - 3x - 1 \right) \\ y' = -x - y \left(x^2 + y^2 - 3x - 1 \right) \end{cases}$$ **(4p)**

Solution. We are going to construct a positively invariant set that does not include any stationary point and conclude using Poincare Bendixson theorem. A positively invariant set is constructed by checking a simple text function $V(x, y) = x^2 + y^2$ and considering it's derivative along trajectories:

$V' = -\left(x^2+y^2\right)\left(x^2+y^2-3x-1\right) = -\left(x^2+y^2\right)\left((x^2-3/2)^2+y^2-3,25\right)$

The expression $\left(x^2+y^2-3x-1\right)$ is negative inside a small enough circle around the origin, for example inside the circle $x^2+y^2=(0.2)^2$ and is positive for x^2+y^2 large enough, for example for $x^2+y^2=4^2$. It implies that the $V'<0$ on the circle $x^2+y^2=4^2$ and negative on the circle

$x^2+y^2=(0.2)^2$ and therefore the ring-shaped set $(0.2)^2 \leq x^2+y^2 \leq 4^2$ is a positively invariant set for the given system.

Stationary points of the system satisfy the system of equations$\begin{cases} 0 = y - x\left(x^2+y^2-3x-1\right) \\ 0 = -x - y\left(x^2+y^2-3x-1\right) \end{cases}$ and therefore satisfy the equation $\left(x^2+y^2\right)\left(x^2+y^2-3x-1\right)=0$. Solutions to this equation are the origin and also points satisfying the equation $\left(x^2+y^2-3x-1\right)=0$. In the second case to be stationary points we observe by checking the system of equations they must also belong to the origin that is impossible. Therefore the only stationary poit of the system is the origin that does not belong to the constructed invariant set.

Therefore by Poincare -Bendixsons theorem the system must have at least on periodic orbit inside the invariant set $(0.2)^2 \leq x^2+y^2 \leq 4^2$.

6. Let $\varphi(t)$ be a solution to the linear system of ODE's $x' = A(t)x$ with periodic coefficients having period T. Show that φ has the property $\varphi(t+T) = k\varphi(t)$ if and only if k is the eigenvalue of the monodromy matrix $\Phi(T) = \exp(TR)$, where $\Phi(t)$ is the fundamental matrix such that $\Phi(0) = I$. **Hint.** Use that any solution $\varphi(t) = \Phi(t)\varphi(0)$. **(4p)**

   Solution. We substitute the expression $\varphi(t) = \Phi(t)\varphi(0)$ into the given equation $\varphi(t+T) = k\varphi(t)$ and arrive to the expression

$$\Phi(t+T)\varphi(0) = k\Phi(t)\varphi(0)$$

   We use the definition of the monodromy matrix: $\exp(TR)$: $\Phi(t+T) = \Phi(t)\exp(TR)$ and conclude that $\Phi(t)\exp(TR)\varphi(0) = k\Phi(t)\varphi(0)$. Multiplying the last relation from the left by $(\Phi(t))^{-1}$ (the fundamental matrix is always non-singular) we get the equation $\exp(TR)\varphi(0) = k\varphi(0)$. This equation means that $\varphi(0)$ is an eigenvalue of the matrix $\exp(TR)$ with eigenvalue k.

   Starting from the statement that the matrix $\Phi(T) = \exp(TR)$ has an eigenvalue k with an eigenvector $\varphi(0)$ and carrying out the same reasoning in the backword order, we arrive to the statement that there is a solution $\varphi(t) = \Phi(t)\varphi(0)$ to the linear system of ODEs $x' = A(t)x$ having the property $\varphi(t+T) = k\varphi(t)$.

   Max. 24 points;

   Threshold for marks: for GU: **VG:** 19 points; **G:** 12 points. For Chalmers: **5:** 21 points; **4:** 17 points; **3**: 12 points;

One must pass both the home assignments and the exam to pass the course. Total points for the course are calculated as $Total = 0.32Assignments + 0.68Exam$ - the average of the points for the home assignments (32%) and for this exam (68%). The same threshold is valid for the exam, for the home assignments, and for the total points for the course.