MATEMATIK	Datum: 2017-01-03	Tid: 08-30 - 12-30
GU, Chalmers	Hjälpmedel: - Inga	
A.Heintz	Telefonvakt: Mattias Lennartsson	Tel.: 5325

Lösningar till **tenta i ODE och matematisk modellering, MMG511, MVE162 (MVE161)**

Answer first those questions that look simpler, then take more complicated ones etc.
Good luck!

1. Formulate general properties of solutions to linear (non-autonomous) systems of ODE: $x' = A(t)x$. Give a proof to the statement about the dimension of the space of all solutions to a linear system of ODE. **(4p)** The set of all solutions to a linear system of ODE with $n \times n$ matrix $A(t)$ possibly dependent on time t is a linear vector space having dimension n, the same as the size of the matrix $A(t)$. **Check the proof in the course book.**

2. Formulate and prove the theorem about instability of equilibrium points to autonomous ODE by Lyapunovs method. **(4p) Check the formulation and the proof in the course book.**

3. Consider the following system of ODE: $\dfrac{d\overrightarrow{r}(t)}{dt} = A\overrightarrow{r}(t)$, with a constant matrix

4. $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$. Give general solution to the system. Find all initial data such that corresponding solutions are bounded. **(4p)**

   Answer. $r = C_1 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + C_2 e^t \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} + C_3 e^t \begin{bmatrix} -1 \\ -t-1 \\ t \end{bmatrix}$

   Solution. $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$, characteristic polynomial: $\lambda^3 - 2\lambda^2 + \lambda = 0$.

   Observe that $\lambda^3 - 2\lambda^2 + \lambda = \lambda(\lambda - 1)^2 = 0$

   Eigenvectors: $v_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \leftrightarrow \lambda_1 = 0$ with simple eigenvalue λ_1; $v_2 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \leftrightarrow \lambda_2 = 1$,

   where λ_2 is a multiple eigenvalue with albebraic multiplicity $n_2 = 2$.

   $A - \lambda_2 I = \begin{bmatrix} -1 & 1 & 1 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$. generalized eigenvector $v_2^{(1)} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ satisfies the equation

   $(A - \lambda_2 I)\, v_2^{(1)} = v_2$ or in matrix form: $\begin{bmatrix} -1 & 1 & 1 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$.

   Corresponding equations are: $\left\{ \begin{array}{rr} -x + y + z = & 0 \\ x = & -1 \\ -x = & 1 \end{array} \right. \Longrightarrow x = -1;\ y = -1;\ z = 0;\ v_2^{(1)} = \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix}$

   For arbitrary initial data $x_0 \in \mathbb{R}^3$, $x_0 = C_1 v_1 + C_2 v_2 + C_3 v_2^{(1)}$ the general solution is expressed as:

   $$x(t) = e^{At} x_0 = C_1 e^{\lambda_1 t} v_1 + C_2 e^{\lambda_2 t} v_2 + \left[I + (A - \lambda_2 I)\, t \right] e^{\lambda_2 t} v_2^{(1)}$$

   Calculate the last term:

$$[I + (A - \lambda_2 I)\, t]\, v_2^{(1)} = \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + t \begin{bmatrix} -1 & 1 & 1 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \right) \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -t+1 & t & t \\ t & 1 & 0 \\ -t & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ -t-1 \\ t \end{bmatrix}$$

Collect all terms and get the answer. Observe that one can multiply any term in the answer with -1 or with any other number, the answer will be still correct. One can get different answers choosing eigenvectors v_1 and v_2 in different ways.

5. Consider the system of ODEs. Find all equilibrium points, investigate their stability and if possible define which type they belong to.

$$\begin{cases} x' = & 4 - 4x - 2y \\ y' = & xy \end{cases} \qquad \textbf{(4p)}$$

Answer: $(1,0)$ saddle point, unstable; $(0,2)$ degenerate knot

Solution. Equilibrium points are $(1,0)$ and $(0,2)$.

Jacoby matrix is $\frac{D}{D(x,y)} \begin{bmatrix} 4-4x-2y \\ xy \end{bmatrix} = \begin{bmatrix} -4 & -2 \\ y & x \end{bmatrix}$

For the point $(1,0)$ Jacobi matrix is equal to $A = \begin{bmatrix} -4 & -2 \\ 0 & 1 \end{bmatrix}$, with eigenvalues $\lambda_1 = -4 < 0$, $\lambda_2 = 1 > 0$. It implies that the equilibrium point is a saddle point and is unstable.

For the point $(0,2)$ the Jacobi matrix is equal to $B = \begin{bmatrix} -4 & -2 \\ 2 & 0 \end{bmatrix}$, with multiple eigenvalue $\lambda_1 = \lambda_2 = -2$. It implies that the equilibrium point is asymptotically stable, it is improper stable node.

6. Show that the following system of ODEs has a periodic solution.

$$\begin{cases} x' = x - 2y - x\left(2x^2 + y^2\right) \\ y' = 4x + y - y\left(2x^2 + y^2\right) \end{cases} \qquad \textbf{(4p)}$$

Solution. Consider the following test function: $V(x,y) = 2x^2 + y^2$. Denoting the right hand side in teh equation by vectorfunction $F(x,y)$ we conclude that

$\nabla V \cdot F = 4x^2 - 8xy - 4x^2\left(2x^2 + y^2\right) + 8xy + 2y^2 - 2y^2(2x^2 + y) = 2\left(1 - (2x^2 + y^2)\right)(2x^2 + y).$

It implies that the elliptic shaped ring: $R = \left\{(x,y) : 0.5 \leq (2x^2 + y) \leq 2\right\}$ is a positive invariant compact set for the ODE, because velocity vectors are directed inside of this ring both on it's outer and inner boundaries ($\nabla V \cdot F < 0$ for $(2x^2 + y) = 2$ and $\nabla V \cdot F > 0$ for $(2x^2 + y) = 0.5$. The origin is the only equilibrium point of the system. One can see it by observing that $V(x,y) = 2x^2 + y^2$ is positive definite and $\nabla V \cdot F(x,y) = 0$ only if $(x,y) = (0,0)$ or if $(2x^2 + y) = 1$.But it is easy to see from the expression for the right hand side for the ODE that in the last case (x,y) cannot be equilibrium point because the right hand side becomes linear with nondegenerate matrix and is zero only in the origin $(x,y) = (0,0)$.

By the Poincare-Bendixson theorem the positively invariant set R not including any equilibrium point must include at least one orbit of a periodic solution.

7. Formulate Banach's contraction principle. Consider the following operator

$$K(x)(t) = \int_0^1 x^2(s)ds + At^2,$$

for all $t \in [0,1]$ acting in the Banach space $C([0,1])$ of continuous functions with norm $\|x\| = \sup_{t\in[0,1]} |x(t)|$.

Find using Banach's contraction principle, conditions on the real constant A such that the operator $K(x)(t)$ has a fixed point. **(4p)**

Solution.

Consider first the $C([0,1])$ norm of the operator $K(x)$.

$$\|K(x)\|_{C([0,1])} = \sup_{t\in[0,1]} \left| \int_0^1 x^2(s)ds + At^2 \right| \leq \left(\|x\|_{C([0,1])} \right)^2 + A.$$

The operator K will have fixed point $y \in C([0,1])$ according to the Banach contraction principle if it would be a contraction in a subset $B \subset C([0,1])$. The last means that for any continuous functions $z(t)$ and $w(t)$ in B the estimate $\|K(z) - K(w)\|_{C([0,1])} < \lambda \|z - w\|_{C([0,1])}$ must be satisfied with a number $\lambda < 1$.

Consider $\|K(z) - K(w)\|_{C([0,1])} = \left\| \int_0^1 z^2(s)ds - \int_0^1 w^2(s)ds \right\|_{C([0,1])} =$

$$\left\| \int_0^1 |z(s) - w(s)| \cdot |z(s) + w(s)| \, ds \right\|_{C([0,1])} \leq$$

$$\left(\sup_{t\in[0,1]} |z(t)| + \sup_{t\in[0,1]} |w(t)| \right) \left(\sup_{t\in[0,1]} |z(t) - w(t)| \right) = \left(\|z\|_{C([0,1])} + \|w\|_{C([0,1])} \right) \|z - w\|_{C([0,1])}$$

It implies that for $\|z\|_{C([0,1])}$ and $\|w\|_{C([0,1])}$ strictly smaller then $1/2$ the norm $\|K(z) - K(w)\|_{C([0,1])} < 1$.

To check conditions of the Banach principle we need to find for wich conditions

the ball $B = \left\{ w \in C([0,1]) : \quad \|w\|_{C([0,1])} < 1/2 \right\}$ in $C([0,1])$ is mapped by the operator K into itself: namely

$\|K(w)\|_{C([0,1])} < 1/2$. The estimate at the beginnign of the consideration shows that it will be valid if $A < 1/4$, because

for $w \in B$ it implies that $\|K(w)\|_{C([0,1])} \leq \left(\|w\|_{C([0,1])} \right)^2 + A \leq 1/4 + A$ for $w \in B$.

Max. 24 points;

Threshold for marks: for GU: **VG:** 19 points; **G:** 12 points. For Chalmers: **5:** 21 points; **4:** 17 points; **3**: 12 points;

One must pass both the home assignments and the exam to pass the course. Total points for the course are calculated as $Total = 0.32 Assignments + 0.68 Exam$ - the average of the points for the home assignments (32%) and for this exam (68%). The same threshold is valid for the exam, for the home assignments, and for the total points for the course.